h-FUNCTION, HILBERT-KUNZ DENSITY FUNCTION AND FROBENIUS-POINCARÉ FUNCTION

CHENG MENG AND ALAPAN MUKHOPADHYAY

Abstract. Given ideals I, J of a noetherian local ring $(R, \mathfrak{m})$ such that $I + J$ is $\mathfrak{m}$-primary and a finitely generated module M, we associate an invariant of (M, R, I, J) called the h-function. Our results on h-function allow extensions of the theories of Frobenius-Poincaré functions and Hilbert-Kunz density functions from the known graded case to the local case, answering a question of Trivedi. When J is $\mathfrak{m}$-primary, we describe the support of the corresponding density function in terms of other invariants of (R, I, J). We show that the support captures the F-threshold: $c^J(I)$, under mild assumptions, extending results of Trivedi and Watanabe. The h-function treats Hilbert-Samuel, Hilbert-Kunz multiplicity and F-threshold on an equal footing. We develop the theory of h-functions in a more general setting which yields a density function for F-signature. A key to many results on h-function is a 'convexity technique' that we introduce, which in particular proves differentiability of Hilbert-Kunz density function almost everywhere, thus contributing to another question of Trivedi.

1. Introduction

Hilbert-Kunz multiplicity and F-signature are numerical invariants appearing in prime characteristics commutative algebra and algebraic geometry. These quantify severity of singularities at a point of a variety and also relate to other invariants, such as the cardinality of the local fundamental group of the punctured spectrum of a strongly F-regular local ring; see [AE08], [CST18] and Section 2. The theory of Hilbert-Kunz multiplicity in the graded case has witnessed two new generalizations in recent years: the Hilbert-Kunz density function and the Frobenius-Poincaré function. Fix a standard graded ring S in prime characteristics and a homogeneous ideal $\mathfrak{a}$ of finite co-length. When the Krull dimension $\dim(S)$ is at least two, Trivedi has proven the existence of a compactly supported real valued continuous function $g_{S,\mathfrak{a}}$ of a real variable- called the Hilbert-Kunz density function- whose integral is the Hilbert-Kunz multiplicity $e_{HK}(\mathfrak{a}, S)$; see Section 2 for details. For the pair $(S, \mathfrak{a})$, where $\dim(S)$ is not necessarily at least two, the associated Frobenius-Poincaré function is an entire function in one complex variable, whose value at the origin is the Hilbert-Kunz multiplicity $e_{HK}(\mathfrak{a}, S)$; see Section 2. These two functions not only encode more subtle invariants of $(S, \mathfrak{a})$ than the Hilbert-Kunz multiplicity but also allow application of geometric tools, such as sheaf cohomology on $\mathrm{Proj}(S)$, and tools from homological algebra. Successful applications of the Hilbert-Kunz density functions have resolved Watanabe and Yoshida's conjecture on the values of Hilbert-Kunz multiplicity of quadric hypersurfaces, rationality of Hilbert-Kunz multiplicities and F-thresholds of two dimensional normal rings among other results; see [Tri21], [TW21], [Tri05], [Tri19].

Building extensions of these two theories to the setting of a noetherian local ring is a natural question; see Trivedi's question [Tri18, Question 1.3]. In this article, we extend the theories of Hilbert-Kunz density function and Frobenius-Poincaré function to the local setting. Our extensions are facilitated by a systematic study of a new function, which

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we call the h-function.

Fix a noetherian local domain $(R,\mathfrak{m})$ of prime characteristic $p>0$ and Krull dimension d, where the Frobenius endomorphism is a finite map. Fix two ideals I,J of R such that $I+J$ is $\mathfrak{m}$-primary. We prove:

Theorem A: Consider the sequence of functions of a real variable

$$h_{n,I,J}(s)=l(\frac{R}{(I^{\lceil sp^n\rceil}+J^{[p^n]})R}),$$

where $J^{[p^n]}$ is the ideal generated by $\{f^{p^n}\mid f\in J\}$; and $l(\_\_)$ is the length function.

(1) (Theorem 3.7, Theorem 3.30) There is a real-valued function of a real variable denoted by $h_{I,J}(s)$ such that given an interval $[s_1,s_2]\subseteq\mathbb{R}$, there is a constant C depending only on s_1,s_2 satisfying

$$|h_{I,J}(s)-\frac{h_{n,I,J}(s)}{p^{nd}}|\leq\frac{C}{p^n},\ \text{for all}\, s\in[s_1,s_2]\,\text{and}\, n\in\mathbb{N}.$$

Consequently, the sequence of functions $\frac{h_{n,I,J}(s)}{p^{nd}}$ converges to $h_{I,J}(s)$ and the convergence is uniform on every compact subset of $\mathbb{R}$.

(2) (Theorem 3.31, Theorem 3.20) Given real numbers $s_2>s_1>0$, there is a constant C'- depending only on s_1,s_2 such that for $x,y\in[s_1,s_2]$,

$$|h_{I,J}(x)-h_{I,J}(y)|\leq C|x-y|.$$

That is, away from zero, $h_{I,J}$ is locally Lipschitz continuous.

The function $h_{I,J}$ is called the h-function associated to the pair (I,J). In fact we prove a version the above theorem for an ideal I and a family of ideals $J_\bullet$ satisfying what we call **Condition C** allowing for applications to other numerical invariants such as F-signature; Theorem 3.7.

Special instances of this h-function have been considered by different authors: in [Tay18] when both I and J are $\mathfrak{m}$-primary, in [BST13] when R is regular, I is principal and $J=\mathfrak{m}$ to study F-signature of a pair and in [Kos17] in the same set up but in a different context. **Theorem A** generalizes their results. Moreover the techniques involved in our proofs yield uniform convergence which is crucial for us.

In Theorem 3.16, we prove that there is a polynomial $P_1(s)$ of degree $\dim(R/J)$ such that $h_{I,J}(s)\leq P_1(s)$ for all s. Using this polynomial bound we prove existence and holomorphicity of a function $F_{R,I,J}(y)$ on the open lower half complex plane; see Theorem 4.3. We moreover show:

$$F_{R,I,J}(y)=\int_{\mathbb{R}}h_{I,J}(t)e^{-ity}(iy)dt.$$

When J is $\mathfrak{m}$-primary, we prove $F_{R,I,J}(y)$ is entire. When $(R,\mathfrak{m},J)$ *comes from a graded pair* $(S,\mathfrak{a})$, i.e. $(R,\mathfrak{m})$ is the localization of a standard graded ring S at the homogeneous maximal ideal, I is the homogeneous maximal ideal and J comes from a homogeneous ideal of finite colength $\mathfrak{a}$, $F_{R,I,J}(y)$ coincides with the Frobenius-Poincaré function of the pair $(S,\mathfrak{a})$; see Proposition 6.8,(3). Unlike [Muk22], our treatment allows us to consider Frobenius-Poincaré function of $(S,\mathfrak{a})$, where $\mathfrak{a}$ need not have finite colength; see Proposition 6.8, (2).

Extending the theory of Hilbert-Kunz density functions is more involved. Set

$$f_n(s) = h_{n,I,J}(s + \frac{1}{p^n}) - h_n(s).$$

When $(R, \mathfrak{m}, J)$ comes from a graded pair $(S, \mathfrak{a})$, where $\dim(S) \geq 2$, we point out that the sequence of functions

$$\frac{f_n(s)}{(p^n)^{d-1}}$$

converges uniformly to the Hilbert-Kunz density function of $(S, \mathfrak{a})$; see Theorem 6.6. But for arbitrary ideals I, J of a local ring $(R, \mathfrak{m})$, the pointwise convergence of , $f_n(s)/(p^n)^{d-1}$ at every s is not clear; in fact when $I = 0$ the sequence does not converge,see Example 5.11. In this direction, we relate the convergence of $f_n(s)/(p^n)^{d-1}$ to the differentiability of $h_{I,J}$ at s. We prove,

If $h_{I,J}(s)$ if differentiable at s, $f_n(s)/(p^n)^{d-1}$ converges to $h'_{I,J}(s)$; see Theorem 5.8

In the direction of differentiability of h, we prove:

Theorem B:(Theorem 5.4,(3),(4)) Let $h_{I,J}$ be as before

(1) The left and right hand derivative of h exist at all non-zero points.
(2) Outside a countable subset of $\mathbb{R}$, h is differentiable; if h is differentiable on an open interval, then it is continuously differentiable on the same interval.

Thm B, (2) implies that for any I, J in the local setting, $f_n(s)/(p^n)^{d-1}$ converges outside a countable subset of $\mathbb{R}$ and coincides with the derivative of $h_{I,J}(s)$; thus outside this countable set the limiting function $f_n(s)/(p^n)^{d-1}$ yields a well-defined notion of density function. In Theorem 5.4, we actually prove existence of density function more generally for a family satisfying **Condition C**. This generalization in particular yields a density function for F-signature. When $(R, \mathfrak{m})$ comes from a graded pair $(S, \mathfrak{a})$ with $\dim(S) \geq 2$, we prove that the corresponding h-function is continuously differentiable and the derivative coincides with the Hilbert-Kunz density function that Trivedi defines. We moreover prove the existence and continuity of the density function to the case when $\mathfrak{a}$ does not have finite colength; see Theorem 6.7. Our work shows that h-function is twice differentiable outside a set of measure zero contributing to Trivedi's question about the order of differentiability of the Hilbert-Kunz density function; see [Tri23, Question 1], Remark 5.5.

Thm B is a consequence of a 'convexity technique' that we introduce. For fixed $s_0 > 0$, in Theorem 5.3, we construct a function $H(s, s_0)$ which we prove to be convex and show that

$$H(s, s_0) = h(s)/c(s) - h(s_0)/c(s_0) + \int_{s_0}^{s} h(t)c'(t)/c^2(t)dt,$$

where $c(s) = s^{\mu-1}/(\mu-1)!$, μ being the cardinality of a set of generators of I. **Thm B** then follows from general properties of convex functions. The underlying idea of the same convexity argument is used to prove Lipschitz continuity of h-functions stated in **Thm A**.

The behaviour of $h_{I,J}$ near zero is more subtle. We prove $h_{I,J}$ is continuous at zero if and only if I is non-zero. In fact our result implies,

Theorem C:(Theorem 8.11) Suppose $\dim(R/I) = d'$. Denote the set of minimal primes of R/I of dimension d' by $\mathrm{Assh}(R/I)$. Then

$$\lim_{s\to 0+} \frac{h(s)}{s^{d-d'}} = \frac{1}{(d-d')!} \sum_{P\in \mathrm{Assh}(R/I)} e_{HK}(J, R/P) e(I, R_P),$$

where $e(I, \_\_)$ denotes the Hilbert-Samuel multiplicity with respect to I. In particular, the order of vanishing $h(s)$ at $s = 0$ is $d - d'$. **Thm C** extends part of [BST13, Thm 4.6],

where R is assumed to be regular, I a principal ideal and $J = \mathfrak{m}$. The h-function treats different numerical invariants of (R, I, J) on an equal footing. When J is $\mathfrak{m}$-primary, for large s, $h_{I,J}(s) = e_{HK}(R, J)$; when I is $\mathfrak{m}$-primary, for $s > 0$ and close to zero $h_{I,J}(s) = e(I, R)\frac{s^d}{d!}$; see [Tay18, Lemma 3.3]. Moreover,

Theorem D:(Theorem 8.6)Suppose J is $\mathfrak{m}$-primary, R is reduced and formally equidimensional (e.g. $(R, \mathfrak{m})$ is a complete domain or localization of a graded domain). Let $\alpha_{R,I,J} = \sup\{s \in \mathbb{R} \mid s > 0\,, h_{I,J}(s) \neq e_{HK}(J, R)\}$. Consider the sequence of numbers,

$$r_I^J(n) = \max\{t \in \mathbb{N} | I^t \nsubseteq (J^{[p^n]})^*\},$$

where $(J^{[p^n]})^*$ denotes the tight closure of the ideal $(J^{[p^n]})$; see Definition 2.5. Then

$$\lim_{n\to\infty} \frac{r_I^J(n)}{p^n} = \alpha_{R,I,J}.$$

We prove, under suitable hypothesis, for e.g. strong F-regularity at every point of $\mathrm{Spec}(R) - \{\mathfrak{m}\}$, $r_I^J(n)/p^n$ in fact converges to the F-threshold $c^J(I)$; see Theorem 8.9. F-threshold is an invariant extensively studied in prime characteristic singularity theory; see [Hun+08b], [MTW05] and is closely related log canonical threshold via reduction modulo p; see [TW04], [HW02]. Whenever $h_{I,J}$ is differentiable, the support of $\frac{d}{ds}h_{I,J}$, which agress with the Hilbert-Kunz density function of (R, I, J), is $[0, \alpha_{R,I,J}]$. This generalizes Trivedi and Watanabe's description of the support Hilbert-Kunz density function which was made when R is strongly F-regular and graded; see Remark 8.8, [TW21, Thm 4.9].

Notation and conventions: All rings are commutative and noetherian. The symbol p denotes a positive prime number. Unless otherwise said, the pair $(R, \mathfrak{m})$ denotes a noetherian local ring R- not necessarily a domain- with maximal ideal $\mathfrak{m}$. By saying $(R, \mathfrak{m})$ is graded, we mean R is a standard graded ring with homogeneous maximal ideal $\mathfrak{m}$. When $(R, \mathfrak{m})$ is assumed to be graded, R-modules and ideals are always assumed to be $\mathbb{Z}$-graded. We assume R has characteristic p and R is F-finite, i.e. the Frobenius endomorphism of R is finite. We index the sequences of numbers and functions by n. Whenever the letter q appears in such a sequence, q denotes p^n. For an ideal $I \subset R$, $I^{[p^n]}$ or $I^{[q]}$ denotes the ideal generated by $\{f^q \mid f \in I\}$ and is called the q or p^n-th *Frobenius power* of I. The operator $l_R(\_\_)$ or simply $l(\_\_)$ denotes the length function. For an R-module M, $F^n_* M$ denotes the R-module whose underlying abelian group is M, but the R-action comes from restriction scalars through the iterated Frobenius morphism $F^n : R \to R$.

2. Background material

Let $(R, \mathfrak{m})$ be a noetherian local or graded ring, J be an $\mathfrak{m}$-primary ideal, M be a finitely generated R-module. Although the germ of Hilbert-Kunz multiplicity was present in Kunz's seminal work [Kun69], its existence was not proven until Monsky's work:

Theorem 2.1. *(see [Mon83]) There is a real number denoted by $e_{HK}(J, M)$ such that,*

$$l(\frac{M}{J^{[p^n]}M}) = e_{HK}(J,M)(p^n)^{\dim(M)} + O((p^n)^{\dim(M)-1}).$$

The number $e_{HK}(J, M)$ is called the Hilbert-Kunz multiplicity of M with respect to J.

Smaller values of $e_{HK}(R, \mathfrak{m})$ predicts milder singularity of $(R, \mathfrak{m})$; see for e.g. [AE08, Cor 3.6], [Man04]. It is imperative to consider Hilbert-Kunz multiplicity with respect to arbitrary ideals, for e.g. to realize F-signature (see Example 3.10)- an invariant characterizing strong F-regularity of $(R, \mathfrak{m})$- in terms of Hilbert-Kunz multiplicity; see [PT18, Cor 6.5]. We refer the readers to [Hun13], [Muk23, Chapter 2] and the references there in for surveying the state of art.

When $(R, \mathfrak{m})$ is graded, Trivedi's Hilbert-Kunz density function refines the notion of Hilbert-Kunz multiplicity:

Theorem 2.2. *(see [Tri18]) Let $(R, \mathfrak{m})$ be graded, J be a finite co-length homogeneous ideal, M be a finitely generated $\mathbb{Z}$-graded R-module. Consider the sequence of functions of a real variable s,*

$$\tilde{g}_{n,M,J}(s) = l([\frac{M}{J^{[q]}M}]_{\lfloor sq \rfloor}).$$

1. *There is a compact subset of $\mathbb{R}$ containing the supports of all $\tilde{g}_n$'s.*
2. *When dim$(M) \geq 1$, there is a function-denoted by $\tilde{g}_{M,J}$- such that $(\frac{1}{q})^{dimM-1}\tilde{g}_{n,M,J}(s)$ converges pointwise to $\tilde{g}_{M,J}(s)$ for all $s \in \mathbb{R}$.*
3. *When dim$(M) \geq 2$, the above convergence is uniform and $\tilde{g}_{M,J}$ is continuous.*
4. $$e_{HK}(J, M) = \int_0^\infty \tilde{g}_{M,J}(s)ds.$$

Definition 2.3. The function $\tilde{g}_{M,J}$ is called the *Hilbert-Kunz density function* of (N, J).

For a graded ring $(R, \mathfrak{m})$, the *Frobenius-Poincaré* function produces another refinement of the Hilbert-Kunz multiplicity. Frobenius-Poincaré functions are essentialy a limiting function of the Hilbert series of $\frac{M}{J^{[q]}M}$ in the variable e^{-iy}, see [Muk22, Rmk 3.6].

Theorem 2.4. *(see [Muk22]) Let M be a finitely generated $\mathbb{Z}$-graded module over a graded $(R, \mathfrak{m})$, J be a finite colength homogeneous ideal. Consider the sequence of entire functions on $\mathbb{C}$*

$$G_{n,M,J}(y) = (\frac{1}{q})^{dim(M)} l([\frac{M}{J^{[q]}M}]_j) e^{-iyj/q}.$$

1. *The sequence of functions $G_{n,M,J}(y)$ converges to an entire function $G_{M,J}(y)$ [1] on $\mathbb{C}$. The convergence is uniform on every compact subset of $\mathbb{C}$.*
2. $$G_{M,J}(0) = e_{HK}(J, M).$$

The last theorem holds for any graded ring which are not necessarily standard graded. For the notion of Hilbert-Kunz density function in the non-standard graded setting, see [TW22]. By [Muk23, Thm 8.3.2], for a standard graded $(R, \mathfrak{m})$ of Krull dimension at least one, the holomorphic Fourier transform of $\tilde{g}_{M,J}$ is $G_{M,J}$, i.e.

$$G_{M,J}(y) = \int_0^\infty \tilde{g}_{M,J}(s)e^{-iys}ds.$$

[1]Note the difference in notation from [Muk22].

Thus when $\dim(M)$ is at least two, the Hilbert-Kunz density function and the Frobenius-Poincaré function determine each other; see [Muk23, Rmk 8.2.4]. Both Hilbert-Kunz density function and Frobenius-Poincaré function capture more subtle graded invariants of (M, J) than the Hilbert-Kunz multiplicity. For e.g. when R is two dimensional, normal, J is generated by forms of the same degree, $\tilde{g}_{R,J}$ and $G_{R,J}$ determine and are determined by slopes and ranks of factors in the Harder-Narasimhan filtration of a syzygy bundle associated to J on $\mathrm{Proj}(R)$; see [Tri05], [Bre07], [Tri18, Example 3.3], [Muk22, Chap 6]. For other results on Hilbert-Kunz density functions and Frobenius-Poincaré functions, see the reference section of [Muk23]. These two functions and the Hilbert-Kunz multiplicity of (R, J) detects J up to its tight closure. Recall:

Definition 2.5. ([HH90, Def 3.1]) Let A be a ring of characteristic $p > 0$. We say $x \in A$ is in the tight closure of an ideal I if there is a c not in any minimal primes of A such that $cx^{p^n} \in I^{[p^n]}$ for all large n. The elements in the tight closure of I form an ideal; denoted I^*.

Theorem 2.6. *Let $I \subseteq J$ be two ideals in $(R, \mathfrak{m})$.*

1. *If $I^* = J^*$, $e_{HK}(I, R) = e_{HK}(J, R)$.*
2. *Conversely, when R is formally equidimensional, i.e. all the minimal primes of the completion $\hat{R}$ have the same dimension, $e_{HK}(I, R) = e_{HK}(J, R)$ implies $I^* = J^*$. When (R, m) is a graded ring where all the minimal primes have the same dimension, $\tilde{g}_{I,R} = \tilde{g}_{J,R}$ or $G_{I,R} = G_{J,R}$ implies $I^* = J^*$.*

3. h-function

Given ideals I, J of a local ring $(R, \mathfrak{m})$ such that $I + J$ is $\mathfrak{m}$-primary and a finitely generated R-module M, we assign a real-valued function $h_{M,I,J}$ of a real variable, which we refer to as the corresponding *h-function*. The existence and continuity of $h_{M,I,J}$ is proven in Section 3.4. When R is additionally a domain and $M = R$, given an ideal I and a family of ideals $\{J_n\}_{n\in\mathbb{N}}$- satisfying what we call **Condition C** below- in Section 3.1, we associate a corresponding h-function which is continuous on $\mathbb{R}_{>0}$.

3.1. h-functions of a domain.

Definition 3.1. Let $\{I_n\}_{n\in\mathbb{N}}$ be a family of ideals of the F-finite local ring R.

1. $I_\bullet$ is called a *weak p-family* if there exists $c \in R$- not contained in any minimal primes of maximal dimension of R such that $cI_n^{[p]} \in I_{n+1}$.
2. $I_\bullet$ is called a *weak p^{-1}-family* if exists a nonzero $\phi \in Hom_R(F_*R, R)$ such that $\phi(F_*I_{n+1}) \subset I_n$.
3. A *big p-family (resp. big p^{-1} family)* is a weak p (resp. p^{-1})-family $I_\bullet$ such that there is an $\alpha \in \mathbb{N}$ for which $\mathfrak{m}^{[p^{n+\alpha}]} \subseteq I_n$ for all n.

A family of ideals where (1) holds with $c = 1$ and $\mathfrak{m}^{[p^n]} \subseteq I_n$, has been called a p-family of ideals; see [HJ18]. Notions of p and p^{-1} families provide an abstract framework for proving existence of asymptotic numerical invariants:

Theorem 3.2. *(see [PT18, Theorem 4.3]) Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of dimension d, $\{I_n\}_{n\in\mathbb{N}}$ a sequence of ideals such that $\mathfrak{m}^{[p^n]} \subset I_e$ for all $n \in \mathbb{N}$.*

1. *If there exists a nonzero $c \in R$ such that $cI_n^{[p]} \subset I_{n+1}$ for all $n \in \mathbb{N}$, then $\eta = \lim_{e\to\infty} 1/p^{nd} l_R(R/I_n)$ exists, and there exists a positive constant C that only depends on c such that $\eta - 1/p^{nd} l_R(R/I_n) \le C/p^n$ for all $n \in \mathbb{N}$.*

(2) *If there exists a non-zero* $\phi \in Hom_R(F_*R, R)$ *such that* $\phi(F_*I_{n+1}) \subset I_n$ *for all* $e \in \mathbb{N}$, *then* $\eta = \lim_{n\to\infty} 1/p^{nd} l_R(R/I_n)$ *exists, and there exists a positive constant* C *that only depends on* ϕ *such that* $1/p^{nd} l_R(R/I_n) - \eta \leq C/p^n$ *for all* $n \in \mathbb{N}$.

(3) *If the conditions in (1) and (2) are both satisfied then there exists a constant* C *that only depends on* c *and* ϕ *such that* $|1/p^{nd} l_R(R/I_n) - \eta| \leq C/p^n$.

Lemma 3.3. *Let* $(R, \mathfrak{m})$ *be a local domain. Let* I_n, J_n *be two weak* p*-families, then so is the family* $I_n + J_n$. *If* I_n, J_n *are two weak* p^{-1}*-families, then so is the family* $I_n + J_n$. *When one of the families are big (*p *or* p^{-1}*), then so is their sum.*

Proof. Suppose there are nonzero elements c_1, c_2 such that $c_1 I_n^{[p]} \subset I_{n+1}$ and $c_2 J_n^{[p]} \subset J_{n+1}$, then $c = c_1c_2$ is still nonzero and satisfies $cI_n^{[p]} \subset I_{n+1}$, $cJ_n^{[p]} \subset J_{n+1}$. So $c(I_n + J_n)^{[p]} \subset I_{n+1} + J_{n+1}$. If there are non-zero elements $\phi_1, \phi_2 \in Hom_R(F_*R, R)$, such that $\phi_1(F_*I_{n+1}) \subset I_n$ and $\phi_2(F_*J_{n+1}) \subset J_n$. For $\phi \in \mathrm{Hom}_R(F_*R, R)$ and $r \in R$, define $r\phi \in \mathrm{Hom}_R(F_*R, R)$ by the formula $r\phi(s) = \phi(rs)$. This puts an R-module structure on $\mathrm{Hom}_R(F_*R, R)$, which turns out to be a torsion free module of rank one. So the R-submodules of $Hom_R(F_*R, R)$ generated by ϕ_1 and ϕ_2 has a nonzero intersection, or in other words, there exist nonzero $c_1, c_2 \in R$ and a nonzero element $\phi \in Hom_R(F_*R, R)$ such that $\phi = \phi_1(F_*(c_1\cdot)) = \phi_2(F_*(c_2\cdot))$. Thus, $\phi(F_*I_{n+1}) \subset I_n$ and $\phi(F_*J_{n+1}) \subset J_n$. So $\phi(F_*(I_{n+1} + J_{n+1})) \subset I_n + J_n$.

To prove the 'big'ness, assume that there is an α such that $\mathfrak{m}^{[p^{n+\alpha}]} \subseteq I_n$. Then we have $\mathfrak{m}^{[p^{n+\alpha}]} \subseteq I_n + J_n$. □

Condition C: Let $(R, \mathfrak{m})$ be an F-finite local ring, I is an ideal and $J_\bullet = \{J_n\}_{n\in\mathbb{N}}$ be a family of ideals in R. We say $I, J_\bullet$ satisfies **Condition C** if

(1) The family $J_\bullet$ is weakly p and also weakly p^{-1}.
(2) For each real number t, there is an α such that $\mathfrak{m}^{[p^{\alpha+n}]} \subseteq I^{\lceil tq \rceil} + J_n$ for all n.

Condition C provides the right framework where we can prove existence of h-functions; see Theorem 3.7.

Definition 3.4. Let $(R, \mathfrak{m})$ be a local or graded ring. Let I be an ideal and $J_\bullet = \{J_n\}_{n\in\mathbb{N}}$ be a family of ideals in R- homogeneous when R is graded, such that $I + J_n$ is $\mathfrak{m}$-primary for all n. For a finitely generated R-module M (homogeneous when R is graded) and $s \in \mathbb{R}$, set

(1) $h_{n,M,I,J_\bullet}(s) = l(\frac{M}{(I^{\lceil sq \rceil} + J_n)M})$.
(2) For an integer d, set

$$h_{n,M,I,J_\bullet,d}(s) = \frac{1}{q^d} l(\frac{M}{(I^{\lceil sq \rceil} + J_n)M}).$$

(3) We denote the limit of the sequence of numbers $h_{n,M,I,J_\bullet,d}(s)$, whenever it exists, by $h_{M,I,J_\bullet,d}(s)$.

Whenever one or more of the parameters $M, I, J_\bullet$ is clear from the context, we suppress those from $h_{n,M,I,J_\bullet}(s)$, $h_{n,M,I,J_\bullet,d}(s)$ or $h_{M,I,J_\bullet,d}(s)$. In the absence of an explicit d, it should be understood that $d = \dim(M)$. When $J_n = J^{[p^n]}$ for some ideal J, $h_{n,M,I,J}, h_{n,M,I,J,d}, h_{M,I,J}$ stand for $h_{n,M,I,J_\bullet}$, $h_{n,M,I,J_\bullet,d}$ and $h_{M,I,J_\bullet,d}$ respectively.

Remark 3.5. (1) With the notational conventions and suppression of parameters declared above, $h_{n,M,I,J}$ stands for both $l(\frac{M}{(I^{\lceil sq \rceil} + J_n)M})$ and $\frac{1}{q^{\dim(M)}} l(\frac{M}{(I^{\lceil sq \rceil} + J_n)M})$. But in the article, it is always clear from the context what $h_{n,M,I,J}$ denotes. So we do not introduce further conventions.

(2) When $(R,\mathfrak{m})$ is graded, M, I and $J_\bullet$ are homogeneous, $h_{n,M,I,J} = h_{n,M_\mathfrak{m},IR_\mathfrak{m},J_\mathfrak{m}}$. So once we prove statements involving h_n's in the local setting, the corresponding statements in the graded setting follow.

The following comparison between ordinary powers and Frobenius powers is used throughout this article:

Lemma 3.6. *Let R be a ring of characteristic $p > 0$, J be an R-ideal generated by μ elements, $k \in \mathbb{N}$, and $q = p^n$ is a power of p. Then $J^{q(\mu+k-1)} \subset (J^{[q]})^k \subset J^{qk}$.*

Proof. The second containment is trivial. We prove the first containment. Let $J = (a_1, ..., a_\mu)$, then $J^{q(\mu+k-1)}$ is generated by $a_1^{u_1}...a_\mu^{u_\mu}$ where $\sum u_i = q(\mu + k - 1)$. Let $a = a_1^{u_1}...a_\mu^{u_\mu}$, $v_i = \lfloor u_i/q \rfloor$ and $b = a_1^{v_1}...a_\mu^{v_\mu}$, then since $qv_i \le u_i$, b^q divides a. Now $qv_i \ge u_i - q + 1$, so $\sum qv_i \ge q(\mu + k - 1) + (-q + 1)\mu = q(k - 1) + \mu > q(k - 1)$, so $\sum v_i \ge k$. This means $b \in J^k$ and $a \in J^{k[q]} = J^{[q]k}$. □

Theorem 3.7. *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of dimension d. Let $J_\bullet$ be a family of ideals such that there is a non-zero $c \in R$ and $\phi \in Hom_R(F_*R, R)$ satisfying $c.J_n^{[p]} \subseteq J_{n+1}$ and $\phi(F_*J_{n+1}) \subseteq J_n$. Let I be an ideal such that for each $s \in \mathbb{R}$, there is an integer α such that $m^{[p^{n+\alpha}]} \subseteq I^{\lceil sq \rceil} + J_n$ for all n. Set $I_n(s) = I^{\lceil sq \rceil} + J_n$.*

1. *Fix $t \in \mathbb{R}$. Choose $\alpha \in \mathbb{N}$ such that $\mathfrak{m}^{[p^{n+\alpha}]} \subseteq I^{\lceil tq \rceil + J_n}$ for all n. Then there exists a positive constant C depending only on c, ϕ, I and α[2] such that for any $s \in (-\infty, t]$,*

$$h_{R,I,J_\bullet,d}(s) = \lim_{n\to\infty} 1/p^{nd} l_R(R/I_n(s)) \textit{ exists, and}$$

$$|1/p^{nd} l_R(R/I_n(s)) - h_{R,I,J_\bullet,d}(s)| \le C/p^n \textit{ for all} n \in \mathbb{N}. \tag{3.1}$$

2. *Given choices $I, J_\bullet$ and $t \in \mathbb{R}$, one can choose C depending only on t, such that Equation (3.1) holds on $[0, t]$.*
3. *On every bounded subset of $\mathbb{R}$, the sequence of functions $h_{n,I,J_\bullet,d}(s)$ converges uniformly to $h_{R,I,J_\bullet}(s)$.*

Proof. (1) When $I = 0$, $I_n(s) = J_n$, so everything follows from Theorem 3.2.

We assume I is non-zero for the rest of the proof. Note $I_n(s)^{[p]} = I^{\lceil sq \rceil [p]} + J_n^{[p]} \subseteq I^{\lceil sq \rceil p} + J_n^{[p]} \subseteq I^{\lceil spq \rceil} + J_n^{[p]}$ as $\lceil sq \rceil p \ge \lceil sqp \rceil$. So

$$c.I_n(s)^{[p]} \subseteq I_{n+1}(s)\ . \tag{3.2}$$

Suppose I is generated by μ-many elements. Then

$$I^{\lceil spq \rceil} \subseteq I^{\lceil sq \rceil p - p} \subseteq I^{[p](\lceil sq \rceil - \mu)};\ \text{ see Lemma 3.6.}$$

Fix a non-zero $r \in (I^\mu)^{[p]}$. Then the last containment implies,

$$\phi(F_*r.F_*I_{n+1}(s)) = \phi(F_*(rI^{\lceil spq \rceil})) + \phi(F_*(rJ_{n+1})) \subseteq \phi(F_*(I^{\lceil sq \rceil [p]})) + J_n \subseteq I_n(s) \text{ for all } s \in \mathbb{R}\ . \tag{3.3}$$

Equation (3.2) and Equation (3.3) imply that, for all s, the non-zero elements $c \in R$ and $\phi(F_*r.\_\_) \in \mathrm{Hom}_R(F_*R, R)$ endow $I_n(s)$ with weakly p and p^{-1} family structures, respectively. The ideal $m^{[p^{n+\alpha}]}$ is contained in $I_n(t)$ and hence in $I_n(s)$ for $s \le t$. The rest follows by applying Theorem 3.2 to the family $I_{n+\alpha}(s)$ for every $s \le t$. The feasibility of choosing C depending only on c, ϕ, α and r also follows from Theorem 3.2. Since $r \in (I^\mu)^{[p]}$ can be chosen depending only on I, the choice of C depends only on c, ϕ, α

[2] In particular C can be chosen independent of the specific choice of $J_\bullet$.

and I.

(2) Once I, J_n satisfying the hypothesis is given and $t \in \mathbb{R}$ is given, c, ϕ, α can be chosen depending only on I, J_n, t.

(3) Every bounded subset of $\mathbb{R}$ is contained in some interval $(-\infty, t]$. The dependence of C only on I, J_n, t and t implies (3). □

The domain assumption is made in the above theorem just so that we can apply Theorem 3.2.

Lemma 3.8. *Suppose I and $J_\bullet$ satisfy the hypothesis of Theorem 3.7. Suppose there is an integer r such that $I^{rp^n} \subseteq J_n$. Then $h_{n,I,J_\bullet}(s)$ and $h_{I,J_\bullet,d}$ are constant on $[r, \infty)$.*

The next two propositions produce examples of an ideal I and ideal family $J_\bullet$ satisfying **Condition C**. For specific choices of $J_\bullet$ and I, the corresponding corresponding functions $h_{I,J_\bullet,d}$ encode widely studied invariants of a prime characteristic ring such as Hilbert-Kunz multiplicity, F-signature, F-threshold. We do not assume R is a domain in the next two examples.

Proposition 3.9. *Let $J_\bullet$ be a family of ideals which is a big p and also p^{-1} family. For any ideal I, I, $J_\bullet$ satisfy **Condition C**.*

Proof. Since $J_\bullet$ is big, there is an α such that $\mathfrak{m}^{[p^{n+\alpha}]} \subseteq J_n$. Thus for every $s \in \mathbb{R}$, $\mathfrak{m}^{[p^{n+\alpha}]} \subseteq I^{\lceil sq \rceil} + J_n$. □

When R is a domain, a big p,p^{-1} family $J_\bullet$ thus produces an h-function. Thanks to Lemma 3.8 such an $h_{I,J_\bullet}$ is eventually constant.

Example 3.10. Examples of $J_\bullet$ which are both big p and also p^{-1} include $J_n = J^{[p^n]}$, where J is an $\mathfrak{m}$-primary ideal. Another example of interest is when J_n is the sequence of ideals defining F-signature of $(R, \mathfrak{m})$ which we now recall. Set $p^\alpha = [k : k^p]$. Take

$$J_n = \{x \in R \,|\, \phi(x) \in \mathfrak{m}, \text{ for all } \phi \in \operatorname{Hom}_R(F^n_* R, R)\}.$$

Then $p^{\alpha n} l(R/I_n)$ coincides with the free rank of $F^n_* R$: the maximal rank of a free module M such that there is an R-module surjection $F^n_* R \to M$; see [Tuc12, Prop 4.5]. The family J_n is both p and p^{-1}; and J_n contains $\mathfrak{m}^{[p^n]}$. Thanks to Theorem 3.2, the limit

$$s(R) := \lim_{n\to\infty} (\frac{1}{q})^{\dim(R)} l(\frac{R}{J_n})$$

exist. The number $s(R)$ measuring the asymptotic growth of the free rank of $F^n_* R$ is called the F-signature of R. The ring $(R, \mathfrak{m})$ is strongly F-regular if and only if $s(R)$ is positive; see [AL03, Thm 0.2]. When R is a domain, for any nonzero ideal I, we have $h_{I,J_\bullet}(s)$ whose value for large s is $s(R)$. The continuity, left-right differentiability of such $h_{I,J_\bullet}$ are consequences of Theorem 5.4.

The examples of h-functions produced by the result below are central to extending theories of Frobenius-Poincaré and Hilbert-Kunz density functions to the local setting.

Proposition 3.11. *For any pair of ideals I, J such that $I + J$ is $\mathfrak{m}$-primary, the ideal I and the family $J_n = J^{[p^n]}$ satisfies **Condition C**.*

Proof. Since $I + J$ is $\mathfrak{m}$-primary, given a real number s, $\mathfrak{m}^{[p^\alpha]} \subseteq I^{\lceil s \rceil} + J$ for some α. Then $m^{[p^{\alpha+n}]} \subseteq (I^{\lceil s \rceil} + J)^{[p^n]} \subseteq I^{\lceil sq \rceil} + J^{[q]}$. So $I^{\lceil sq \rceil} + J^{[q]}$ is a big p and p^{-1} family. □

For two $\mathfrak{m}$-primary ideals I, J, in [Tay18] Taylor considers s-multiplicity(function) which is a scalar multiple of the corresponding $h_{I,J}$. When $J_n = J^{[q]}$, our proof of the existence of h function in Theorem 3.7 is not only different from the proof of Theorem 2.1 of [Tay18], but also is still valid when both I and J are not necessarily $\mathfrak{m}$-primary. Moreover, in Theorem 3.7, the flexibility of choosing C depending only ϕ and c is a byproduct of our proof; this flexibility is crucial in Theorem 3.13 and later.

3.2. **Growth of h-function, $\mathfrak{m}$-adic continuity.** Next, we investigate how $h_{n,I,J_\bullet}(s)$ changes when the I or $J_\bullet$ is replaced by another ideal or ideal family which is $\mathfrak{m}$-adically close the initial one. The results we prove are used later in Section 6, for example, to prove continuity of Hilbert-Kunz density function $\tilde{g}_{M,J}$ for non $\mathfrak{m}$-primary J; see Theorem 6.7.

Lemma 3.12. *Let R be a noetherian local ring, I, J be two R-ideals such that $I+J$ is $\mathfrak{m}$-primary. Let I', J' be two ideals such that $I \subset I'$, $J \subset J'$. Then $h_{n,M,I,J}(s) \geq h_{n,M,I',J'}(s)$.*

Proof. If $I \subset I'$, $J \subset J'$ then $(I^{\lceil tp\rceil} + J^{[p]})M \subset (I'^{\lceil tp\rceil} + J'^{[p]})M$, so $l(M/(I^{\lceil tp\rceil} + J^{[p]})M) \geq l(M/(I'^{\lceil tp\rceil} + J'^{[p]})M)$, which just means $h_{n,M,I,J}(s) \geq h_{n,M,I',J'}(s)$. □

Theorem 3.13. *Let $(R, \mathfrak{m})$ be a noetherian local ring. Assume $I, J_\bullet$ satisfy **Condition C**.*
(1) Fix $s_0 \in \mathbb{R}$. We can choose t depending only on $I, J_\bullet, s_0$ such that for any ideals $J \subset \mathfrak{m}^t$,$I \subset I'$, and all n,

$$h_{n,M,I',J_\bullet}(s) = h_{n,M,I',J_\bullet+J^{[p^n]}}(s) \text{ for } s \leq s_0.$$

(2) Assume $J_\bullet$ is both big p and p^{-1} family. There exists a constant c such that for any ideals $I' \subset \mathfrak{m}^t$, $t \in \mathbb{N}$ and $s \in \mathbb{R}$,

$$h_{n,M,I,J}(s - c/t) \leq h_{n,M,I+I',J}(s) \leq h_{n,M,I,J}(s) \leq h_{n,M,I+I_t,J}(s + c/t).$$

(3) Fix $s_0 > 0$. There exists a t_0 and a constant c, both only depending on $s_0, I, J_\bullet$ such that for any $t \geq t_0$, $I_t \subseteq \mathfrak{m}^t$,

$$h_{n,M,I,J_\bullet}(s - c/t) \leq h_{n,M,I+I_t,J_\bullet}(s) \leq h_{n,M,I,J_\bullet}(s) \leq h_{n,M,I+I_t,J_\bullet}(s + c/t),$$

for $s \leq s_0$.

Proof. (1) Let t be the smallest integer such that $\mathfrak{m}^{t[q]} \subset I^{\lceil s_0 q\rceil} + J_n$ for all n. By the previous lemma, it suffices to consider the case where $J = \mathfrak{m}^t$. So for $I \subseteq I'$,

$$I'^{\lceil sq\rceil} + J_n = I'^{\lceil sq\rceil} + J_n + \mathfrak{m}^{t[q]} \text{ for } s \leq s_0 \text{ and all } n \in \mathbb{N},$$

proving the desired statement.

(2) Since $J_\bullet$ is a big family, we can choose t_0 such that $\mathfrak{m}^{t_0[q]} \subseteq J_n$ for all n. We may also assume $I' = \mathfrak{m}^t$. Let $\mathfrak{m}$ be generated by μ-elements, set $\epsilon_t = t_0\mu/t$. Then $\mathfrak{m}^{t\lceil \epsilon_t q\rceil} \subseteq \mathfrak{m}^{t_0\mu q} \subseteq \mathfrak{m}^{t_0[q]} \subset J_n$ for all n. So

$$(I + \mathfrak{m}^t)^{\lceil sq\rceil} = \sum_{0\leq j\leq \lceil sq\rceil} I^{\lceil sq\rceil - j}\mathfrak{m}^{tj} \subset I^{\lceil sq\rceil - \lceil \epsilon_t q\rceil} + \mathfrak{m}^{t\lceil \epsilon_t q\rceil} \subset I^{\lceil sq\rceil - \lceil \epsilon_t q\rceil} + J_n \subseteq I^{\lceil (s - t_0\mu/t)q\rceil} + J_n$$

Thus we have

$$l(M/(I^{\lceil (s-t_0\mu/t)q\rceil} + J_n)M) \leq l(M/((I + \mathfrak{m}^t)^{\lceil sq\rceil} + J_n)M) \leq l(M/(I^{\lceil sq\rceil} + J_n)M).$$

So taking $c = t_0\mu$ verifies the first two inequalities. These equalities are independent of s, so we may replace s by $s + c/t$ to get the third inequality.

(3) By (1) we can choose t_1 depending on $s_0, I, J_\bullet$ such that $h_{n,M,I',J_\bullet+\mathfrak{m}^{t_1[q]}}(s) = h_{n,M,I',J_\bullet}(s)$ whenever $I \subset I'$ and $s \leq s_0+1$. By (2), we can choose c depending on $J+\mathfrak{m}^{t_1}$ such that

$$h_{n,M,I,J_\bullet+\mathfrak{m}^{t_1[q]}}(s-c/t) \leq h_{n,M,I+I_t,J_\bullet+\mathfrak{m}^{t_1[q]}}(s) \leq h_{n,M,I,J+J_\bullet+\mathfrak{m}^{t_1[q]}}(s) \leq h_{n,M,I+I_t,J_\bullet+\mathfrak{m}^{t_1[q]}}(s+c/t),$$

for $I_t \subseteq m^t$. Take $t_0 = c$. Since for $t \geq t_0$ and $s \leq s_0$, $s + \frac{c}{t} \leq s_0+1$, the above chain of inequalities imply

$$h_{n,M,I,J_\bullet}(s-c/t) \leq h_{n,M,I+I_t,J_\bullet}(s) \leq h_{n,M,I,J_\bullet}(s) \leq h_{n,M,I+I_t,J_\bullet}(s+c/t).$$

□

Assertion (1) of the theorem above allows us to replace $J_\bullet$ by a big p and p^{-1} family in questions involving local structure of h-functions. This observation is repeatedly used later; see Theorem 6.7.

Next we prove that the sequence $h_{n,I,J_\bullet,d}(s)$ is uniformly bounded on every compact subset. When $J_\bullet = J^{[p^n]}$ for some J, we refine the bound to show that $h_{n,I,J_\bullet,d}(s)$ is bounded above by a polynomial of degree $\dim(\frac{R}{J})$ in Theorem 3.16. The uniform (in n) polynomial bound on h_n is used in the extension of the theory of Frobenius-Poincaré functions in Lemma 4.1, Theorem 4.3.

Lemma 3.14. *In a local ring $(R,\mathfrak{m})$, let $I, J_\bullet$ satisfy **condition C**. Let M be a finitely generated R-module. Given $s_0 \in \mathbb{R}$, there is a constant C depending only on s_0 such that*

$$h_{n,M,I,J_\bullet}(s) \leq C.q^d \textit{ for all} n.$$

Proof. Choose α such that $\mathfrak{m}^{[p^{n+\alpha}]} \subseteq I^{\lceil s_0 q\rceil} + J_n$. So for $s \leq s_0$,

$$h_{n,M,I,J_\bullet}(s) \leq l(\frac{M}{\mathfrak{m}^{[p^{n+\alpha}]}}M) \leq Cq^d.$$

The last ineuqality is a consequence of [Mon83]. □

Remark 3.15. Given a noetherian local ring $(R,\mathfrak{m},k)$ containing $\mathbb{F}_p$, a field extension $k \subseteq L$ denote by S the $\mathfrak{m}$-adic completion of $L \otimes_k \hat{R}$. Here $\hat{R}$ is the $\mathfrak{m}$-adic completion of R and $\hat{R}$ can be treated as a k-algebra thanks to the existence of coefficient field of $\hat{R}$; see [Sta23, tag 0323]. The residue field of the local ring S is isomorphic to L. The natural map $R \to S$ is faithfully flat. Now given a finite length R-module M, $l_R(M) = l_S(S \otimes_R M)$. We use this observation to make simplifying assumption on the residue field of R.

Theorem 3.16. *Let $(R,\mathfrak{m},k)$ be a noetherian local ring of dimension d, I, J be two R-ideals such that $I+J$ is $\mathfrak{m}$-primary. Assume I is generated by μ elements, M is generated by ν elements, and $d' = \dim R/J$. Then:*

1. *There exist a polynomial $P_1(s)$ of degree d' such that for any $s \geq 0$,*

$$\frac{l(M/I^{\lceil sq\rceil} + J^{[q]}M)}{l(R/\mathfrak{m}^{[q]})} \leq P_1(s).$$

   Moreover if $d' > 0$, the leading coefficient of P_1 can be taken to be $\frac{\nu e(I,R/J)}{d'!}$

2. *There exist a polynomial $P_2(s)$ such that*

$$\frac{l(M/I^{\lceil sq\rceil} + J^{[q]}M)}{q^d} \leq P_2(s).$$

   In other words, $h_{n,M,d}(s)/q^d \leq P_2(s)$.

3. *There exists a polynomial P_3 of degree d' and leading coefficient $\frac{\nu e(I,R/J)e_{HK}(R)}{d'!}$ such that for any $s \geq 0$,*

$$\overline{\lim}_{n\to\infty}\frac{l(M/I^{\lceil sq\rceil} + J^{[q]}M)}{q^d} \leq P_3(s).$$

Proof. We may assume that the residue field is perfect by using Remark 3.15

(1) Suppose M is generated by ν many elements. Then

$$\begin{aligned} l(M/I^{\lceil sq\rceil} + J^{[q]}M) &\leq \nu l(R/I^{\lceil sq\rceil} + J^{[q]}) \\ &\leq \nu l(R/(I^{\lceil s\rceil})^{[q]} + J^{[q]}) \\ &\leq \nu l(F_*^n R/(I^{\lceil s\rceil} + J)F_*^n R) \\ &\leq \nu \mu_R(F_*^n R) l(R/I^{\lceil s\rceil} + J) \end{aligned}$$

Let P_0 be the Hilbert-Samuel polynomial for the I-adic filtration on R/J; P_0 has degree d' and leading coefficient $\frac{\nu e(I,R/J)}{d'!}$. Fix s_0 such that $l(R/I^{\lceil s\rceil}+J) = P_0(\lceil s\rceil)$ and P_0 is non-decreasing for $s \geq s_0$. Thus for $s \geq s_0$,

$$l(R/I^{\lfloor s\rfloor} + J) \leq P_0(s+1).$$

When $\frac{R}{J}$ has Krull dimension zero, $P_0(s) = l(R/J)$ and $l(R/I^{\lceil s\rceil} + J) \leq P_0(s+1)$ for all s, so we can take the desired P_1 to be $P_0(s+1)$. When R/J has positive Krull dimension, we can add a suitable positive constant to $P_0(s+1)$ to get a P_1 so that $l(R/I^{\lfloor s\rfloor} + J) \leq P_1(s)$ on $[0,2]$ and thus on $\mathbb{R}$.

(2) Since $\lim_{n\to\infty} l(R/\mathfrak{m}^{[q]})/q^d$ exists,

$$C = \sup_n l(R/\mathfrak{m}^{[q]})/q^d$$

exists. So for any n, $l(R/\mathfrak{m}^{[q]})/q^d \leq C$, and $P_2 = CP_1$ satisfies (2).

(3)

$$\begin{aligned} &\overline{\lim}_{n\to\infty}\frac{l(M/I^{\lceil sq\rceil} + J^{[q]}M)}{q^d} \\ \leq\ &\overline{\lim}_{n\to\infty}\frac{l(M/I^{\lceil sq\rceil} + J^{[q]}M)}{l(R/\mathfrak{m}^{[q]})}\overline{\lim}_{n\to\infty}\frac{l(R/\mathfrak{m}^{[q]})}{q^d} \\ &\leq e_{HK}(R)P_1(s). \end{aligned}$$

So $P_3 = e_{HK}(R)P_1$ works.

□

3.3. **Lipschitz continuity of h-functions, application of a 'convexity technique'.** Proving continuity of $h_{R,I,J_\bullet}$- when R is a domain is more involved than proving its existence. In this subsection, we develop results aiding the proof of Lipschitz continuity of $h_{R,I,J_\bullet}$; see Theorem 3.20. When $J_n = J^{[q]}$, these results are used to prove existence and continuity of the h-function of a finitely generated module in Theorem 3.30, by reducing the problem to the case where R is reduced. The key result aiding these applications is Theorem 3.19. We prove this by utilizing the monotonicity of a certain numerical function. This 'convexity technique' is repeatedly used later to prove left and right differentiability of the h-function in among other properties. The required monotonicity result appears in Lemma 3.17. This is an adaptation and generalization of Boij-Smith's result in [BS15] which is suitable for our purpose.

Lemma 3.17. *Let $(R,\mathfrak{m})$ be a noetherian local ring, I be an $\mathfrak{m}$-primary ideal generated by μ elements, M be a finitely generated R-module, S be the polynomial ring of μ-variables over $\frac{R}{\mathfrak{m}}$. Then the function $i \to l(I^iM/I^{i+1}M)/l(S_i)$ is decreasing for $i \geq 0$.*

Proof. Consider the associated graded ring $gr_I(R)$. Since I is generated by a set of μ elements, as a graded ring $gr_I(R)$ is a quotient of the standard graded polynomial ring $R/I[T_1,...,T_\mu]$ over R/I. Recall $S = \frac{R}{\mathfrak{m}}[T_1,...,T_\mu]$. Since M/IM is Artinian, there exists a filtration

$$0 = N_0 \subset N_1 \subset ... \subset N_l = M/IM, \text{ such that } N_{j+1}/N_j = \frac{R}{\mathfrak{m}} \text{ for } 0 \leq j \leq l-1.$$

Let M_j be the $gr_I(R)$-submodule of $gr_I(M)$ spanned by N_j. Then M_{j+1}/M_j is annihilated by $\mathfrak{m}gr_I(R)$. So it is naturally a $gr_I(R)/\mathfrak{m}gr_I(R)$-module, hence is an S-module, and it is generated in degree 0. So by Theorem 1.1 of [BS15], for any $i \geq 0$,

$$l(M_{j+1}/M_j)_i/l(S_i) \leq l(M_{j+1}/M_j)_{i+1}/l(S_{i+1}).$$

Since truncation at degree i is an exact functor from $gr_I(R)$-modules to R-modules, taking sum over $0 \leq j \leq l-1$ we get $l(M_l)_i/l(S_i) \leq l(M_l)_{i+1}/l(S_{i+1})$. Since $M_l = gr_I(R)N_l = gr_I(M)$, we are done. ☐

When I is a principal ideal, the above lemma manifests into the following easily verifiable result.

Example 3.18. Let R be a noetherian local ring, f be an element in R such that R/fR has finite length. Then for any $j \geq i$, $l(f^iR/f^{i+1}R) \geq l(f^jR/f^{j+1}R)$. This means that the function $i \to l(R/f^iR)$ is convex on $\mathbb{N}$; see Definition 5.2.

Theorem 3.19. *Let R be a noetherian local ring, M be a finitely generated module of dimension d. Suppose $I, J_\bullet$ satisfy* ***Condition C***. *Fix $0 < s_1 < s_2 < \infty \in \mathbb{R}$. Then there is a constant C and a power $q_0 = p^{n_0}$ that depend on s_1, s_2, but independent of n such that for any $s_1 \leq s \leq s_2 - 1/q$ and $q \geq q_0$*

$$l(\frac{(I^{\lceil sq \rceil} + J_n)M}{(I^{\lceil sq \rceil+1} + J_n)M}) \leq Cq^{d-1}$$

In other words, whenever $s_1 \leq s \leq s_2 - 1/q$ and $q \geq q_0$,

$$|h_{n,M}(s + 1/q) - h_{n,M}(s)| \leq Cq^{d-1}.$$

Proof. We may assume $s_1, s_2 \in \mathbb{Z}[1/p]$. Otherwise, since $\mathbb{Z}[1/p]$ is dense in $\mathbb{R}$, we can choose $s'_1 \in (0, s_1) \cap \mathbb{Z}[1/p]$, $s'_2 \in (s_2, \infty) \cap \mathbb{Z}[1/p]$ and replace s_1, s_2 by s'_1, s'_2. Choose $s_3 \in \mathbb{Z}[1/p]$ such that $0 < s_3 < s_1$ and choose q_0 such that $s_1q_0, s_2q_0, s_3q_0 \in \mathbb{Z}$. Let I be generated by a set of μ many elements. Applying Lemma 3.17 to the module M/J_nM we know for any $0 \leq t \leq \lceil sq \rceil$,

$$\frac{l(\frac{I^{\lceil sq \rceil}(M/J_nM)}{I^{\lceil sq \rceil+1}(M/J_nM)})}{\binom{\mu+\lceil sq \rceil-1}{\mu-1}} \leq \frac{l(\frac{I^t(M/J_nM)}{I^{t+1}(M/J_nM)})}{\binom{\mu+t-1}{\mu-1}}.$$

Rewritten, the above inequality yields

$$\frac{l(\frac{(I^{\lceil sq \rceil}+J_n)M}{(I^{\lceil sq \rceil+1}+J_n)M})}{\binom{\mu+\lceil sq \rceil-1}{\mu-1}} \leq \frac{l(\frac{(I^t+J_n)M}{(I^{t+1}+J_n)M})}{\binom{\mu+t-1}{\mu-1}}.$$

Thus for $s_1 \leq s \leq s_2 - \frac{1}{q}$,

$$\begin{aligned}(\lceil sq\rceil - s_3q)l(\frac{(I^{\lceil sq\rceil}+J_n)M}{(I^{\lceil sq\rceil+1}+J_n)M}) &\leq \binom{\mu+\lceil sq\rceil-1}{\mu-1}\sum_{t=s_3q}^{\lceil sq\rceil-1}\frac{l(\frac{(I^t+J_n)M}{(I^{t+1}+J_n)M})}{\binom{\mu+t-1}{\mu-1}}\\ &\leq \frac{\binom{\mu+\lceil sq\rceil-1}{\mu-1}}{\binom{\mu+s_3q-1}{\mu-1}}l(\frac{(I^{\lceil sq\rceil}+J_n)M}{(I^{s_3q}+J_nM})\\ &\leq \frac{\binom{\mu+\lceil sq\rceil-1}{\mu-1}}{\binom{\mu+s_3q-1}{\mu-1}}[l(\frac{M}{(I^{\lceil sq\rceil}+J_n)M})-l(\frac{M}{(I^{s_3q}+J_n)M})]\\ &\leq \frac{\binom{\mu+s_2q-1}{\mu-1}}{\binom{\mu+s_3q-1}{\mu-1}}[l(\frac{M}{(I^{s_2q}+J_n)M})-l(\frac{M}{(I^{s_3q}+J_n)M})].\end{aligned}$$

Therefore for $s_1 \leq s \leq s_2 - \frac{1}{q}$ and $q \geq q_0$,

$$l(\frac{(I^{\lceil sq\rceil}+J_n)M}{(I^{\lceil sq\rceil+1}+J_n)M}) \leq \frac{1}{s_1q-s_3q}\frac{\binom{\mu+s_2q-1}{\mu-1}}{\binom{\mu+s_3q-1}{\mu-1}}[l(\frac{M}{(I^{s_2q}+J_n)M})-l(\frac{M}{(I^{s_3q}+J_n)M})] \leq Cq^{d-1}.$$

By Lemma 3.14, we can choose a constant C' depending only on s_2 such that for $s \leq s_2$,

$$l(\frac{M}{(I^{sq}+J_n)M}) \leq C'q^d.$$

Since $\binom{\mu+s_2q-1}{\mu-1}/\binom{\mu+s_3q-1}{\mu-1}$ is bounded above by a constant depending on s_1, s_3 and s_3 depends only on s_2, we can choose C depending only on s_1, s_2 such that for all n and $q \geq q_0$,

$$l(\frac{(I^{\lceil sq\rceil}+J_n)M}{(I^{\lceil sq\rceil+1}+J_n)M}) \leq Cq^{d-1}.$$

Here C is a constant only depending on s_1, s_2, s_3, and s_3 depends only on s_1. □

Therefore, whenever $h_{M,I,J_\bullet}$ exists, it is locally Lipschitz continuous away from zero.

Theorem 3.20. *Let I be an ideal and $J_\bullet$ be a family of ideals satisfying **Condition C** in a domain $(R, \mathfrak{m})$ of Krull dimension d. Given real numbers $0 < s_1 < s_2$, there is a constant C depending only in s_1, s_2 such that for any $x, y \in [s_1, s_2]$,*

$$|h_R(x) - h_R(y)| \leq C|x-y|$$

Proof. Given s_1, s_2 as above and x, y in $[s_1, s_2]$, by Theorem 3.19, we can choose a constant C depending only on s_1, s_2 such that

$$|h_{n,R}(x) - h_{n,R}(y)| = |h_{n,R}(\frac{\lceil qx\rceil}{q}) - h_{n,R}(\frac{\lceil qy\rceil}{q})| \leq C|\frac{\lceil qx\rceil}{q} - \frac{\lceil qy\rceil}{q}|q^d \text{ for all } n.$$

Divide both sides by q^d and take limit as n approaches infinity. Since for any real number s, $\frac{h_n(s)}{q^d}$ and $\lceil qs\rceil/q$ converge to $h_R(s)$ and s respectively,

$$|h_R(x) - h_R(y)| \leq C|x-y|.$$

□

Lemma 3.21. *Assume the residue field of R is perfect and M is a module of dimension d. For each integer $n_0 \geq 0$ and fixed $0 < s_1 < s_2 < \infty \in \mathbb{R}$, there is a constant C independent of n such that*

$$|h_{n+n_0,M,I,J}(s) - h_{n,F_*^{n_0}M,I,J}(s)| \leq Cq^{d-1}$$

for any $s_1 \leq s \leq s_2$.

Proof. For any q_0, $\lceil sqq_0 \rceil \leq \lceil sq \rceil q_0 \leq \lceil sqq_0 \rceil + q_0$. We have,

$$\begin{aligned}
&|h_{n+n_0,M,I,J}(s) - h_{n,F_*^{n_0}M,I,J,d}(s)| \\
&= |l(M/(I^{\lceil sqq_0 \rceil} + J^{[qq_0]})M) - l(F_*^{n_0}M/(I^{\lceil sq \rceil} + J^{[q]})F_*^{n_0}M)| \\
&= |l(M/(I^{\lceil sqq_0 \rceil} + J^{[qq_0]})M) - l(M/(I^{\lceil sq \rceil [q_0]} + J^{[qq_0]})M)| \\
&= (l(I^{\lceil sqq_0 \rceil} + J^{[qq_0]})M/(I^{\lceil sq \rceil q_0} + J^{[qq_0]})M) + l(I^{\lceil sq \rceil q_0} + J^{[qq_0]})M/(I^{\lceil sq \rceil [q_0]} + J^{[qq_0]})M)) .
\end{aligned}$$

Note that $1/q_0 \lceil sqq_0 \rceil \geq sq \geq \lceil sq \rceil - 1$, so $\lceil sq \rceil q_0 \leq \lceil sqq_0 \rceil + q_0$, so $I^{\lceil sqq_0 \rceil + q_0} \subset I^{\lceil sq \rceil q_0}$. Suppose I is generated by μ elements, then by Lemma 3.6, $I^{\lceil sq \rceil q_0} \subset I^{(\lceil sq \rceil - \mu + 1)[q_0]}$. Now by Theorem 3.19, we can choose a constant C depending only on s_1, s_2 such that for all $s \in [s_1, s_2]$,

$$\begin{aligned}
&l(\frac{(I^{\lceil sqq_0 \rceil} + J^{[qq_0]})M}{(I^{\lceil sq \rceil q_0} + J^{[qq_0]})M}) + l(\frac{(I^{\lceil sq \rceil q_0} + J^{[qq_0]})M}{(I^{\lceil sq \rceil [q_0]} + J^{[qq_0]})M}) \\
&\leq l(\frac{(I^{\lceil sqq_0 \rceil} + J^{[qq_0]})M}{(I^{\lceil sqq_0 \rceil + q_0} + J^{[qq_0]})M}) + l(\frac{(I^{(\lceil sq \rceil - \mu + 1)[q_0]} + J^{[qq_0]})M}{(I^{\lceil sq \rceil [q_0]} + J^{[qq_0]})M}) \leq Cq^{d-1} .
\end{aligned}$$

□

The lemma above allows us to replace M by $F_*^{n_0}M$. Since we may replace R by $R/\text{ann}F_*^{n_0}M$ and for large enough n_0, $\text{ann}F_*^{n_0}M$ contains the nilradical of R; case, we may assume R is reduced while proving the existence of $h_{M,I,J}$.

Corollary 3.22. Assume the residue field of R is perfect. For each $n_0 \geq 0$, $h_{M,I,J,d}(s)$ exists if and only if $h_{F_*^{n_0}M,I,J,d}(s)$ exists, and if they both exist then

$$q_0^d h_{M,I,J,d}(s) = h_{F_*^{n_0}M,I,J,d}(s).$$

3.4. **Existence of $h_{M,I,J}$.** For a noetherian local ring $(R, \mathfrak{m})$, R-ideals I, J such that $I + J$ is $\mathfrak{m}$-primary and a finitely generated module, we prove the existence of $h_{M,I,J}$ in Theorem 3.30. We prove preparatory results to reduce this problem to the situation where $M = R$ and R is a domain. We prove the local Lipschitz continuity of $h_{M,I,J}$ in Theorem 3.31. Recall:

Definition 3.23. Set $\text{Assh}R = \{P \in \text{Spec}R : \dim R = \dim R/P\}$.

Lemma 3.24. *[Mon83, Proof of Lemma 1.3]If M, N are two R-modules such that $M_P \cong N_P, \forall P \in \text{Assh}R$. Then there is an exact sequence*

$$0 \to N_1 \to M \to N \to N_2 \to 0$$

such that $\dim N_1, \dim N_2 \leq$ *dim*$(R) - 1$. *Moreover it breaks up into two short exact sequences:*

$$0 \to N_1 \to M \to N_3 \to 0$$
$$0 \to N_3 \to N \to N_2 \to 0$$

such that dim$(N_3) <$ *dim*(R).

Lemma 3.25. *Let $N \subset M$ be two R-modules of finite length, and take $a \in R$, then $l(M/aM) \geq l(N/aN)$.*

Proof. Consider the commutative diagram,

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & 0:_N a & \longrightarrow & N & \xrightarrow{a} & N & \longrightarrow & \frac{N}{aN} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0:_M a & \longrightarrow & M & \xrightarrow{a} & M & \longrightarrow & \frac{M}{aM} & \longrightarrow & 0 \end{array}$$

We see the map $0 :_N a \to 0 :_M a$ is injective. By the additivity of length on short exact sequences we see $l(M/aM) = l(0 :_M a) \geq l(0 :_N a) = l(N/aN)$. □

Lemma 3.26. *Let M_1, M_2, M_3, M_4 be four submodules of an R-module M such that $M_3 \subset M_1$, $M_4 \subset M_2$. Then $M_1 + M_2/M_3 + M_4$ has a filtration with factors which are quotients of M_1/M_3 and M_2/M_4. In particular, if M_1/M_3 and M_2/M_4 have finite lengths then so does $M_1 + M_2/M_3 + M_4$ and $l(M_1 + M_2/M_3 + M_4) \leq l(M_1/M_3) + l(M_2/M_4)$.*

Proof. Consider the filtration

$$0 \subseteq \frac{M_3 + M_2}{M_3 + M_4} \subseteq \frac{M_1 + M_2}{M_3 + M_4}$$

The factors in the above filtration, namely $M_3 + M_2/M_3 + M_4$ and $M_1 + M_2/M_3 + M_2$, are quotients of M_2/M_4 and M_1/M_3 respectively. □

Lemma 3.27. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Suppose $I, J_\bullet$ satisfy **condition C**, and M is a module of dimension $d' \leq d - 1$. Fix $s_0 \in \mathbb{R}$. Then there are constants C_1, C_2 depending on s_0 but independent of n such that $l(\mathrm{Tor}_0^R(R/(I^{\lceil sq \rceil} + J_n), M)) \leq C_1 q^{d-1}$ and $l(\mathrm{Tor}_1^R(R/(I^{\lceil sq \rceil} + J_n), M)) \leq C_2 q^{d-1}$ for any $s \leq s_0$. Moreover if $J_\bullet$ is big, C_1, C_2 can be chosen independent of s.*

Proof. Since $I, J_\bullet$ satisfy **Condition C**, we can find an $\mathfrak{m}$-primary ideal J such that for $s \leq s_0$, $J^{[q]} \subseteq I^{\lceil sq \rceil} + J_n$ for all n. As $M/J^{[q]}M$ surjects onto $\mathrm{Tor}_0^R(R/(I^{\lceil sq \rceil} + J_n), M)$, and we can find a constant C_1, such that $l(M/J^{[q]}M) \leq C_1 q^{\dim M}$, $l(\mathrm{Tor}_0^R(R/I^{\lceil sq \rceil} + J^{[q]}, M)) \leq C_1 q^{d-1}$.

To see the bound on Tor_1, for a fixed $s \leq s_0$, consider the exact sequence:

$$0 \to (I^{\lceil sq \rceil} + J_n)/J^{[q]} \to R/J^{[q]} \to R/(I^{\lceil sq \rceil} + J^{[q]}) \to 0$$

So by the long exact sequence of Tor, it suffices to show that we can choose C_2 satisfying

$$l(\mathrm{Tor}_1^R(R/J^{[q]}, M)) \leq C_2 q^{d-1} \text{ and } l(\frac{I^{\lceil sq \rceil} + J_n}{J^{[q]}} \otimes M) \leq C_2 q^{d-1}.$$

Choosing a C_2 satisfying the first inequality is possible thanks to [HMM04, Lemma 1.1]. For the remaining inequality, by taking a prime cyclic filtration of M, we may assume $M = R/P$ for some $P \in \mathrm{Spec}(R)$ with $\dim M \leq \dim R - 1$. In this case, $P \notin \mathrm{Assh}(R)$. So we can choose $b \in P$ such that $\dim R/bR \leq \dim R - 1$. Taking $M = R/J^{[q]}$ and $N = I^{\lceil sq \rceil} + J^{[q]}/J^{[q]}$ in Lemma 3.25, we see that we can enlarge C_2 independently of s and q so that

$$\begin{aligned} l(l(\frac{I^{\lceil sq \rceil} + J_n}{J^{[q]}} \otimes_R R/P) &\leq l(l(\frac{I^{\lceil sq \rceil} + J_n}{J^{[q]}} \otimes_R R/bR) \\ &\leq l(R/J^{[q]} \otimes_R R/bR) = l(R/bR + J^{[q]}) \leq C_2 q^{d-1}. \end{aligned}$$

So we are done. □

Lemma 3.28. *Let M, N be two finitely generated R-modules that are isomorphic at $P \in \mathrm{Assh}R$. Then for any $t > 0$, there is a constant C depending on M, I, J, t but independent of n such that for any $s < t$*

$$|h_{n,M,d}(s) - h_{n,N,d}(s)| \leq C/q$$

Moreover if J is $\mathfrak{m}$-primary, then C can be chosen independently of t.

Proof. By Lemma 3.24, there is an exact sequence

$$0 \to N_1 \to M \to N \to N_2 \to 0$$

such that $\dim N_1, \dim N_2 \le d-1$. And it breaks up into two short exact sequences:

$$0 \to N_1 \to M \to N_3 \to 0$$

$$0 \to N_3 \to N \to N_2 \to 0$$

Now by the long exact sequence of Tor we get

$$|l(M/(I^{\lceil sq\rceil} + J^{[q]})M) - l(N_3/(I^{\lceil sq\rceil} + J^{[q]})N_3)| \le l(N_1/(I^{\lceil sq\rceil} + J^{[q]})N_1)$$

$$|l(N_3/(I^{\lceil sq\rceil}+J^{[q]})N_3)-l(N/(I^{\lceil sq\rceil}+J^{[q]})N)| \le l(N_2/(I^{\lceil sq\rceil}+J^{[q]})N_2)+l(\mathrm{Tor}_1^R(R/(I^{\lceil sq\rceil}+J^{[q]}), N_2))$$

Thus by Lemma 3.27, there is a constant C such that

$$|l(M/(I^{\lceil sq\rceil} + J^{[q]})M) - l(N/(I^{\lceil sq\rceil} + J^{[q]})N)| \le Cq^{d-1}$$

□

Lemma 3.29. *Let $(R, \mathfrak{m}, k)$ be a local ring, I, J be two ideals such that $I + J$ is $\mathfrak{m}$-primary, and M be a finitely generated R-module. For any $0 < s_1 < s_2 < \infty$, there is a constant C depending on M, I, J, s_1, s_2 but independent of n such that for any $s_1 \le s \le s_2$*

$$|h_{n+1,M,d}(s) - h_{n,M,d}(s)| \le C/q$$

Proof. We may assume that the residue field is perfect by using Remark 3.15. Choose sufficiently large n_0 such that $R/annF_*^{n_0}M$ is reduced. The positive constants C_1, C_2, C_3 chosen below depends only on M, I, J, s_1, s_2 and is independent of n. By Lemma 3.21,

$$|h_{n+n_0,M,I,J}(s) - h_{n,F_*^{n_0}M,I,J}(s)| \le C_1q^{d-1}$$

and

$$|h_{n+n_0+1,M,I,J}(s) - h_{n+1,F_*^{n_0}M,I,J}(s)| \le C_1q^{d-1}$$

So it suffices to prove existence of a suitable C such that

$$|h_{n+1,F_*^{n_0}M,d}(s) - h_{n,F_*^{n_0}M,d}(s)| \le C/q.$$

Replacing M by $F_*^{n_0}M$ and R by $R/annF_*^{n_0}M$, so we may assume R is reduced. In this case,

$$|h_{n+1,M,I,J}(s) - h_{n,F_*M,I,J}(s)| \le C_2q^{d-1}.$$

Thanks to the reducedness of R, the localizations of $M^{\oplus p^d}$ and F_*M are isomorphic at all $P \in AsshR$. So by Lemma 3.28,

$$|h_{n,F_*M,I,J}(s) - p^dh_{n,M,I,J}(s)| \le C_3q^{d-1}.$$

Thus one can choose a C which depends only on M, I, J, s_1, s_2 such that for all $s \in [s_1, s_2]$ and $n \in \mathbb{N}$,

$$|h_{n+1,M,I,J}(s) - p^dh_{n,M,I,J}(s)| \le Cq^{d-1}.$$

Dividing by $(pq)^d$, we get

$$|h_{n+1,M,I,J,d}(s) - h_{n,M,I,J,d}(s)| \le C/q.$$

□

Theorem 3.30. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, I, J be two R-ideals such that $I + J$ is $\mathfrak{m}$-primary, and M is a finitely generated R-module. Then for every $s \in \mathbb{R}$,*

$$\frac{1}{q^{dim(M)}} \lim_{n\to\infty} h_{n,M,I,J}(s) = h_{M,I,J}(s)$$

exists. Moreover the convergence is uniform on $[s_1, s_2]$ for any $0 < s_1 < s_2 < \infty$.

Proof. By replacing R by $R/\mathrm{ann}(M)$, we may assume $\dim(M) = \dim(R)$. Given s_1, s_2 as in the statement, it follows from Lemma 3.29 that $h_{n,M,I,J}(s)/q^{\dim(M)}$ is uniformly Cauchy on $[s_1, s_2]$. So the theorem follows. □

We also have:

Theorem 3.31. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d, I, J be two R-ideal, $I + J$ is $\mathfrak{m}$-primary, and M be a finitely generated R-module. Then:*

1. *$h_M(s)$ is Lipschitz continuous on $[s_1, s_2]$ for any $0 < s_1 < s_2 < \infty$. Consequently, it is continuous on $(0, \infty)$.*
2. *$h_M(s)$ is increasing. It is 0 on $(-\infty, 0]$. It is continuous if and only if it is continuous at 0, if and only if $\lim_{s\to 0^+} h_M(s) = 0$. The limit $\lim_{s\to 0^+} h_M(s)$ always exists and is nonnegative.*
3. *Assume J is $\mathfrak{m}$-primary. Then for $s >> 0$, $h_{n,M}(s) = e_{HK}(J, M)$ is a constant.*
4. *There is a polynomial $P(s)$ of degree $\dim R/J$ such that $h_M(s) \le P(s)$ on $\mathbb{R}$.*

Proof.
(1) An argument similar to that in the proof of Theorem 3.20 with R replaced by M and $J_n = J^{[q]}$ yields a proof. The difference is that when $J_n = J^{[q]}$, we know the existence of $h_{M,I,J}$.

(2) If $s_1 \le s_2$, then $\lceil s_1 q\rceil \le \lceil s_2 q\rceil$, so $I^{\lceil s_2 q\rceil} \subset I^{\lceil s_1 q\rceil}$. This implies
$$l(M/(I^{\lceil s_1 q\rceil} + J^{[q]})M) \le l(M/(I^{\lceil s_2 q\rceil} + J^{[q]})M),$$
which is just
$$h_{n,M}(s_1) \le h_{n,M}(s_2).$$
So after dividing $p^{n\dim M}$ and let $n \to \infty$, we get $h_M(s_1) \le h_M(s_2)$. This implies $h_M(s)$ is increasing; so in particular the limit $\lim_{s\to 0^+} h_M(s)$ always exists and is at least $h_M(0)$. If $s \le 0$, then $\lceil sq\rceil \le 0$, so $I^{\lceil sq\rceil} = R$. Thus $M/(I^{\lceil s_1 q\rceil} + J^{[q]})M = 0$ and $h_{n,M}(s) = 0$ for any n, so $h_M(s) = 0$. So $h_M(s)$ is continuous on $(-\infty, 0)$ and $(0, \infty)$, and $\lim_{s\to 0^-} h_M(s) = 0 = h_M(0)$, so we get (2).

(3) Let J be generated by μ elements. For $s >> 0$, $I^{\lfloor s/\mu\rfloor} \subset J$. So $I^{\lceil sq\rceil} \subset I^{\lfloor s/\mu\rfloor q\mu} \subset J^{q\mu} \subset J^{[q]}$, so $h_{n,M}(s) = l(M/J^{[q]}M)$ and $h_M(s) = \lim_{n\to\infty} \frac{l(M/J^{[q]}M)}{q^d} = e_{HK}(J, M)$. If $s = 0$ then $I^{\lceil sq\rceil} = R$ so $h_{n,M}(0) = 0$.

(4) This is a corollary of Theorem 3.16 and Theorem 3.30.

□

The associativity formula for h-function below follows directly from Lemma 3.28.

Proposition 3.32. *Let M be a d-dimensional finitely generated R-module. Let $P_1, P_2, \ldots, P_t$ be the d-dimensional minimal primes in the support of M. Then,*
$$h_{M,I,J,d}(s) = \sum_{j=1}^{t} l_{R_{P_j}}(M_{P_j}) h_{R/P_j, IR/P_j, JR/P_j, d}(s).$$

4. Frobenius-Poincaré function in the local setting

We prove the existence of Frobenius-Poincaré functions in the local setting. Given an ideal I and a family $J_\bullet$ and a finitely generated R-module M, set
$$f_{n,M,I,J_\bullet}(s) = h_{n,M,I,J_\bullet}(s + \frac{1}{q}) - h_{n,M,I,J_\bullet}(s).$$
When $J_n = J^{[q]}$, $f_{n,M,I,J}(s)$ represents $f_{n,M,I,J_\bullet}(s)$. We drop one or more parameters in $f_{n,M,I,J_\bullet}$ when there is no resulting confusion. For the rest of this article, we denote the

imaginary part a complex number y by $\Im y$ and the open lower half complex plane by Ω, i.e. $\Omega = \{y \in \mathbb{C} \mid \Im y < 0\}$.

Lemma 4.1. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d, I, J be two R-ideal, $I + J$ is $\mathfrak{m}$-primary, and M be a finitely generated R-module. Consider the function defined by the infinite series*

$$F_{n,M,I,J}(y) := \sum_{j=0}^{\infty} f_{n,M,I,J}(j/q)e^{-iyj/q}$$

Then $F_{n,M,I,J}(y)$ defines a holomorphic function on Ω. We often drop one or more parameters in $F_{n,M,I,J}$ when there is no chance of confusion.

Proof. There is a polynomial P such that $f_{n,M}(s) \leq h_{n,M}(s+1) \leq P(s)$ for any s; see Theorem 3.16, Theorem 3.31, assertion (2). Thus

$$|f_{n,M,R,I,J}(j/q)e^{-iyj/q}| \leq P(j/q)e^{j\Im y/q}.$$

Since for fixed $\epsilon > 0$, the series $\sum_{0 \leq j < \infty} P(j/q)e^{-j\epsilon/q}$ converges, on the region where $\Im y < -\epsilon$, the sequence of functions $\sum_{j=0}^{\infty} f_{n,M,R,I,J}(j/q)e^{-iyj/q}$ converges uniformly. The limit function is thus holomorphic [Ahl79, Thm 1, Chap 5]. Taking union over all $\epsilon > 0$, we see $F_{n,M}(y)$ exists and is holomorphic on Ω. □

Remark 4.2. For a big p, p^{-1} family $J_\bullet$, the analogous $F_{n,M,I,J_\bullet}(y)$ defined using $f_{n,M,I,J_\bullet}$ is entire since the corresponding sum is a finite sum.

Now, we want to check the convergence of $(F_{n,M,I,J}(y)/q^{\dim(M)})_n$ whenever it exists. We will be repeatedly using the dominated convergence: if a sequence of measurable functions f_n converges to f pointwise on a measurable set Σ and there is a measurable function g such that $|f_n| \leq g$ on Σ for any n and $\int_\Sigma |g| < \infty$, then $\int_\Sigma |f_n - f|$ converges to 0, so in particular $\int_\Sigma f_n$ converges to $\int_\Sigma f$.

Theorem 4.3. *Let $(R, \mathfrak{m}, k)$ be a local ring, I, J be two R-ideal, $I + J$ is $\mathfrak{m}$-primary, and M be a finitely generated R-module of dimension d.*
(1) Assume J is $\mathfrak{m}$-primary. Then $F_{M,I,J}(y) = \lim_{n\to\infty} F_{n,M}(y)/p^{n \dim M}$ exists for all $y \in \mathbb{C}$. This convergence is uniform on any compact set of $\mathbb{C}$. Suppose $h_M(s)$ is constant for $s \geq C$, then $F_{M,I,J}(y) = \int_0^C h_M(t)iye^{-iyt}dt + h_M(C)e^{-iyC}$.
(2) Assume J is not necessarily $\mathfrak{m}$-primary. Then for every $y \in \Omega$, $F_{n,M}(y)/p^{n \dim M}$ converges to

$$F_{M,I,J}(y) = \int_0^\infty h_M(t)e^{-iyt}iydt.$$

Moreover, this convergence is uniform on any compact subset of Ω and $F_M(y) := F_{M,I,J}(y)$ is holomorphic on Ω.

Proof.

(1) Since J is $\mathfrak{m}$-primary, then $h_M(s) = h_M(C)$ for some fixed $C > 0$ and any $s \geq C$; see Lemma 3.8 and Proposition 3.32. Then,

$$\begin{aligned} F_{n,M}(y) &= \sum_{j=0}^{\infty} f_{n,M}(j/q)e^{-iyj/q} \\ &= \sum_{j=0}^{\infty}(h_{n,M}((j+1)/q) - h_{n,M}(j/q))e^{-iyj/q} \\ &= \sum_{j=0}^{Cq-1}(h_{n,M}((j+1)/q) - h_{n,M}(j/q))e^{-iyj/q} \\ &= \sum_{j=0}^{Cq-1} h_{n,M}(j/q)(e^{-iy(j-1)/q} - e^{-iy(j)/q}) + h_{n,M}(C)e^{-iy(C-\frac{1}{q})} \\ &= \sum_{j=0}^{Cq-1} h_{n,M}(j/q)e^{-iyj/q}(e^{iy/q} - 1) + h_{n,M}(C)e^{-iy(C-\frac{1}{q})} \\ &= \int_0^C h_{n,M}(t)e^{-iy\lceil tq\rceil/q}q(e^{iy/q} - 1)dt + h_{n,M}(C)e^{-iy(C-\frac{1}{q})} \ . \end{aligned}$$

Fix a compact subset K of $\mathbb{C}$. Given $\delta > 0$, choose $b > 0$ such that for all $y \in K$, $t \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\int_0^b (\frac{1}{q^d}|h_{n,M}(t)e^{-iy\lceil tq\rceil/q}q(e^{iy/q} - 1)| + |h_M(t)e^{-iyt}(iy)|)dt \leq \frac{\delta}{2}.$$

We have

$$|\frac{1}{q^d}F_{n,M}(y) - \int_0^C h_M(t)e^{-iyt}(iy)dt - h_M(C)e^{-iyC}|$$

$$\leq \int_0^C |h_{n,M,d}(t)e^{-iy\lceil tq\rceil/q}q(e^{iy/q} - 1) - h(y)iye^{-iyt}|dt + |h_{n,M}(C)e^{-iy(C-\frac{1}{q})} - h_M(C)e^{-iyC}|$$

$$\leq \int_0^b (|h_{n,M,d}(t)e^{-iy\lceil tq\rceil/q}q(e^{iy/q} - 1)| + |h_M(t)e^{-iyt}(iy)|)dt$$

$$+ \int_b^C |h_{n,M,d}(t)e^{-iy\lceil tq\rceil/q}q(e^{iy/q} - 1) - h(y)iye^{-iyt}|dt + |h_{n,M}(C)e^{-iy(C-\frac{1}{q})} - h_M(C)e^{-iyC}|.$$

Moreover for $y \in K$, there is a constant C' independent of n such that for all $t \in [b, C]$

$$|h_{n,M,d}(\lfloor tq\rfloor/q) - h_M(t)| \leq C'/q \,\text{and}\, |e^{-iy\lfloor tq\rfloor/q}q(e^{iy/q} - 1) - e^{iyt}(iy)| \leq C'/q.$$

Thus we can choose N_0 such that for all $n \geq N_0$ and $y \in K$,

$$|\frac{1}{q^d}F_{n,M}(y) - \int_0^C h_M(t)e^{-iyt}(iy)dt - h_M(C)e^{-iyC}| \leq \delta.$$

This proves the desired uniform convergence.

(2)We prove uniform convergence of $F_{n,M}/q^{\dim(M)}$ to the integral on every compact subset of Ω; the holomorphicity of F_M is then a consequence of [Ahl79, Thm1, Chap 5]. We have

$$\begin{aligned}
F_{n,M}(y) &= \sum_{j=0}^{\infty} f_{n,M}(j/q)e^{-iyj/q} \\
&= \sum_{j=0}^{\infty}(h_{n,M}((j+1)/q) - h_{n,M}(j/q))e^{-iyj/q} \\
&= \sum_{j=0}^{\infty} h_{n,M}(j/q)(e^{-iy(j-1)/q} - e^{-iy(j)/q}) \\
&= \sum_{j=0}^{\infty} h_{n,M}(j/q)e^{-iyj/q}(e^{iy/q} - 1) \\
&= \int_0^{\infty} h_{n,M}(t)e^{-iy\lceil tq\rceil/q}q(e^{iy/q} - 1)dt \ .
\end{aligned}$$

The rearrangements leading to the second and third equality are possible thanks to the absolute convergences implied by Theorem 3.16. Fix any compact $K \subseteq \Omega$. Using triangle inequality, we get

$$\begin{aligned}
&|h_{n,d}(t)e^{-iy\frac{\lfloor tq\rfloor}{q}}q(e^{iy/q} - 1) - h(t)e^{-iyt}(iy)| \\
\leq\ &|h_{n,d}(t) - h(t)||e^{-iy\frac{\lceil tq\rceil}{q}}q(e^{iy/q} - 1)| + |h(t)||e^{-iy\frac{\lceil tq\rceil}{q}} - e^{-iyt}||q(e^{iy/q} - 1)| \\
&+|h(t)||e^{-iyt}||q(e^{iy/q} - 1) - iy| \\
=\ &|h_{n,d}(t) - h(t)||e^{-iy\frac{\lceil tq\rceil}{q}}q(e^{iy/q} - 1)| + |h(t)e^{-iyt}||e^{-iy(\frac{\lceil tq\rceil}{q}-t)} - 1||q(e^{iy/q} - 1)| \\
&+|h(t)e^{-iyt}||q(e^{iy/q} - 1) - iy| \ .
\end{aligned}$$

It follows from the power series expansion of e^z at zero and the boundedness of K that there are constants C_1, C_2 such that for all $y \in K$, $t \in \mathbb{R}$ and $n \in \mathbb{N}$

$$|q(e^{iy/q} - 1)| \leq C_1|y|, \ |q(e^{iy/q} - 1) - iy| \leq C_2\frac{|y|^2}{q}, \ |e^{-iy(\frac{\lceil tq\rceil}{q}-t)} - 1| \leq C_1|y(\frac{\lceil tq\rceil}{q} - t)|.$$

Choose $\epsilon > 0$ such that $K \subseteq \{y \in \mathbb{C} \,|\, \Im y < -\epsilon\}$. Using the comparisons above, we get for all $y \in K$, $t \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&|h_{n,d}(t)e^{-iy\frac{\lceil tq\rceil}{q}}q(e^{iy/q} - 1) - h(t)e^{-iyt}(iy)| \\
\leq\ &|h_{n,d}(t) - h(t)|e^{-\epsilon t}C_1|y| + |h(t)e^{-\epsilon t}|C_1^2|y|^2|\frac{\lceil tq\rceil}{q} - t| + |h(t)e^{\epsilon t}|C_2\frac{|y|^2}{q} \\
\leq\ &|h_{n,d}(t) - h(t)|e^{-\epsilon t}C_1|y| + |h(t)e^{-\epsilon t}|C_1^2\frac{|y|^2}{q} + |h(t)e^{-\epsilon t}|C_2\frac{|y|^2}{q} \ .
\end{aligned}$$

Taking integral on $\mathbb{R}_{\geq 0}$, we get for $y \in K$ and all $n \in \mathbb{N}$

$$\begin{aligned}
&|\frac{1}{q^d}F_{n,M}(y) - F_{M,I,J}(y)| \\
\leq\ &C_1|y|\int_0^{\infty}|h_{n,d}(t) - h(t)|e^{-\epsilon t}dt + (C_1^2 + C_2)\frac{|y|^2}{q}\int_0^{\infty}|h(t)|e^{-\epsilon t}dt \ .
\end{aligned}$$

Thanks to Theorem 3.16, (2), we can choose a polynomial $P_2 \in \mathbb{R}[t]$ such that $|h_{n,d}(t)| \leq |P_2(t)|$ for all n and $t \in \mathbb{R}$. Since $|P_2(t)e^{-\epsilon t}|$ is integrable on $\mathbb{R}_{\geq 0}$, by dominated convergence

$$\lim_{n\to\infty} \int_0^\infty |h_{n,d}(t) - h(t)|e^{-\epsilon t}dt = 0.$$

Using this in the last inequality implies uniform convergence of $\frac{1}{q^d}F_{n,M}(y)$ to $F_{M,I,J}(y)$ on K. ☐

Remark 4.4. Suppose $h_M(y)$ is constant for $y \geq C$. Since for $y \in \Omega$, $h_M(C)e^{-iyC}$ converges to zero as y approaches infinity, the two descriptions of h_M in this case match on Ω. When $J_\bullet$ is both big p and p^{-1}, our argument actually produces a corresponding entire function $F_{M,I,J_\bullet}(y)$.

Definition 4.5. Let I, J be two ideals in $(R, \mathfrak{m})$ such that $I + J$ is $\mathfrak{m}$-primary. For a finitely generated R-module M, the function $F_{M,I,J}(y)$ is called the *Frobenius-Poincaré function* of (M, I, J).

We drop one or more parameters from $F_{M,I,J}$ when there is no possible source of confusion.

The next result directly follows from Proposition 3.32.

Corollary 4.6. Let M, N be two R-modules such that their localization are isomorphic at all $P \in \mathrm{Assh}R$. Then $F_M(y) = F_N(y)$.

Proof. This is true because $h_M(s) = h_N(s)$. ☐

5. Differentiability of h-function, density function in the local setting

In this section, we discuss the extension of the theory of Hilbert-Kunz density function in the local setting.

Definition 5.1. Let I be an ideal and $J_\bullet$ be a family of ideals in $(R, \mathfrak{m})$ satisfying **Condition C**. For a finitely generated R-module M and $s \in \mathbb{R}$, recall

$$f_{n,M,I,J_\bullet}(s) = h_{n,M,I,J_\bullet}(s + \frac{1}{q}) - h_{n,M,I,J_\bullet}(s) = l(\frac{(I^{\lceil sq \rceil} + J_n)M}{(I^{\lceil sq \rceil + 1} + J_n)M}).$$

Whenever $((\frac{1}{p^n})^{\dim(M)-1} f_{n,M,I,J_\bullet}(s))_n$ converges, we call the limit the *density function* of $(M, I, J_\bullet)$ at s and denote the limit by $f_{M,I,J_\bullet}(s)$. Whenever $f_{M,I,J_\bullet}(s)$ exists for all $s \in \mathbb{R}$, the resulting function $f_{M,I,J_\bullet}$ is called the *density function* of $(M, I, J_\bullet)$.

We often drop one or more parameters from $f_{n,M,I,J_\bullet}(s), f_{M,I,J_\bullet}(s), f_{M,I,J_\bullet}$ whenever those are clear from the context.

In Theorem 5.8, we relate the existence of $f_{M,I,J_\bullet}(s)$ to the differentiability of $h_{M,I,J_\bullet}$ at s-whenever $h_{M,I,J_\bullet}$ exists. We show that $h_{M,I,J_\bullet}$ is always left and right differentiable everywhere on the real line. The new ingredient is our 'convexity technique'. The h-function being Lipschitz continuous is differentiable outside a set of measure zero. But our method shows that the h-function is differentiable outside a countable set. Recall:

Definition 5.2. Let S be a subset of $\mathbb{R}$. We call a function $\lambda : S \to \mathbb{R}$ to be *convex* if for elements of S, $s_1 < s_2 \leq t_1 < t_2$,

$$\frac{\lambda(s_2) - \lambda(s_1)}{s_2 - s_1} \geq \frac{\lambda(t_2) - \lambda(t_1)}{t_2 - t_1}.$$

Convexity is a notion that appears naturally in mathematical analysis. For references on convex functions, see [NP06].
Let $I, J_\bullet, M$ be as above. Now we lay the groundwork for the construction of the convex function $\mathcal{H}(s, s_0)$ in Theorem 5.3. Fix μ such that I is generated by μ-many elements. Set $M_q = M/J_nM$ and S to be the polynomial ring in μ many variables over $R/\mathfrak{m}$. Given a compact interval $[a, b] \subseteq (0, \infty)$, thanks to Theorem 3.19 we can choose C such that for all $x \in [a, b]$ and $n \in \mathbb{N}$

$$\frac{I^{\lceil xq\rceil}M_q}{I^{\lceil xq\rceil+1}M_q} = h_n(x+\frac{1}{q}) - h_n(x) \leq Cq^{\dim M-1}.$$

Recall,

$$l(S_{\lceil xq\rceil}) = \binom{\mu+\lceil xq\rceil-1}{\mu-1} = 1/(\mu-1)!(\lceil xq\rceil)^{\mu-1} + O(\lceil xq\rceil^{\mu-2}).$$

Fix $s_0 \in \mathbb{R}$. Taking cues from these two estimates, for $s > s_0$ we define

$$\mathcal{H}_n(s, s_0) = \sum_{j=\lceil s_0q\rceil}^{\lceil sq\rceil-1} q^{\mu-\dim(M)-1} l(I^jM_q/I^{j+1}M_q)/l(S_j)\ . \tag{5.1}$$

Theorem 5.3. *Let $I, J_\bullet$ in the local ring $(R, \mathfrak{m})$ satisfy* ***Condition C****, M be a finitely generated R-module of Krull dimsnion d, I be generated by a set of μ elements. Set $M_q = M/J_nM$, fix $s_0 \in \mathbb{R}_{>0}$. Consider the two situations:*

(A) *R is a domain and $M = R$.*
(B) *$J_n = J^{[q]}$ for some ideal J such that $I + J$ is $\mathfrak{m}$-primary and M is any finitely generated R-module.*

Set $c(s) = \frac{s^{\mu-1}}{(\mu-1)!}$. In the context of (A) or (B)[3]*, set*

$$\mathcal{H}(s, s_0) = h_{M,I,J_\bullet}(s)/c(s) - h_{M,I,J_\bullet}(s_0)/c(s_0) + \int_{s_0}^{s} h_{M,I,J_\bullet}(t)c'(t)/c^2(t)dt.$$

(1) *On any compact subset of (s_0, ∞), $\mathcal{H}_n(s, s_0)$ uniformly converges to $\mathcal{H}(s, s_0)$.*
(2) *The function $\mathcal{H}(s, s_0)$ is a convex function on (s_0, ∞).*

Proof. (1) We have

$$\begin{aligned}
\mathcal{H}_n(s, s_0) &= \sum_{j=\lceil s_0q\rceil}^{\lceil sq\rceil-1} q^{\mu-d-1} l(I^jM_q/I^{j+1}M_q)/l(S_j) \\
&= \sum_{j=\lceil s_0q\rceil}^{\lceil sq\rceil-1} q^{\mu-d-1}(l(M_q/I^{j+1}M_q) - l(M_q/I^jM_q))/l(S_j) \\
&= q^{\mu-d-1}l(M_q/I^{\lceil sq\rceil}M_q)/l(S_{\lceil sq\rceil-1}) - q^{\mu-d-1}l(M_q/I^{\lceil s_0q\rceil}M_q)/l(S_{\lceil s_0q\rceil}) \\
&\quad + \sum_{j=\lceil s_0q\rceil+1}^{\lceil sq\rceil-1} q^{\mu-d-1}l(M_q/I^jM_q)(1/l(S_{j-1}) - 1/l(S_j))\ .
\end{aligned}$$

Since we are in the context of (A) or (B), $q^{\mu-d-1}l(M_q/I^{\lceil sq\rceil}M_q)/l(S_{\lceil sq\rceil-1})$ converges to $h(s)/c(s)$ and $q^{\mu-d-1}l(M_q/I^{\lceil s_0q\rceil}M_q)/l(S_{\lceil s_0q\rceil})$ converges to $h(s_0)/c(s_0)$. Also,

[3] $h_{M,I,J_\bullet}$ exists in the context of (A) or (B)

$$\sum_{j=\lceil s_0q\rceil+1}^{\lceil sq\rceil-1} q^{\mu-d-1}l(M_q/I^jM_q)(1/l(S_{j-1})-1/l(S_j))$$
$$=\int_{s_0}^{s-1/q}\frac{l(M_q/I^{\lceil tq\rceil}M_q)}{q^d}(\frac{1}{l(S_{\lceil tq\rceil-1})}-\frac{1}{l(S_{\lceil tq\rceil})})(q^\mu)dt\ .$$

When q approaches infinity, $\frac{l(M_q/I^{\lceil tq\rceil}M_q)}{q^d}$ converges to $h_M(t)$, and $(\frac{1}{l(S_{\lceil tq\rceil-1})}-\frac{1}{l(S_{\lceil tq\rceil})})(q^\mu)$ converges to $c'(t)/c^2(t)$. Also, all these convergence are uniform on any compact subset of $(0,\infty)$. So we get a uniform convergence (uniform on s) on any compact subset of (s_0,∞):

$$\int_{s_0}^{s-1/q}\frac{l(M_q/I^{\lfloor tq\rfloor}M_q)}{q^d}(\frac{1}{l(S_{\lfloor tq\rfloor-1})}-\frac{1}{l(S_{\lfloor tq\rfloor})})(q^\mu)dt$$
$$\to\int_{s_0}^{s}h(t)c'(t)/c^2(t)dt.$$

This proves that $\mathcal{H}_n(s,s_0)$ converges to $\mathcal{H}(s,s_0)$ and the convergence is uniform on any compact subset of (s_0,∞).

(2) We claim $\mathcal{H}_n$ is convex on $1/p^n\mathbb{Z}\cap(s_0,\infty)$. To this end, it suffices to show

$$\mathcal{H}_n(\frac{i+1}{p^n},s_0)-\mathcal{H}_n(\frac{i}{p^n},s_0)\geq\mathcal{H}_n(\frac{i+2}{p^n},s_0)-\mathcal{H}_n(\frac{i+1}{p},s_0).$$

By definition, this is equivalent to showing

$$l(I^iM_q/I^{i+1}M_q)/l(S_i)\geq l(I^{i+1}M_q/I^{i+2}M_q)/l(S_{i+1}),$$

which follows from Lemma 3.17. This convexity of $\mathcal{H}_n(s,s_0)$ implies the convexity of the limit function $\mathcal{H}(s,s_0)$ on $(s_0,\infty)\cap\mathbb{Z}[1/p]$. Therefore for $s_1<s_2\leq t_1<t_2$ in $(s_0,\infty)\cap\mathbb{Z}[1/p]$,

$$\frac{H(s_2,s_0)-H(s_1,s_0)}{s_2-s_1}\geq\frac{H(t_2,s_0)-H(t_1,s_0)}{t_2-t_1}.$$

Since $\mathcal{H}(s,s_0)$ is continuous on (s_0,∞), $(s,t)\to H(t,s_0)-H(s,s_0)/(t-s)$ is continuous. Moreover as $\mathbb{Z}[1/p]\cap(s_0,\infty)$ is dense in (s_0,∞), the slope inequality defining a convex function (see Definition 5.2) holds for $\mathcal{H}(s,s_0)$ for points in (s_0,∞). □

Theorem 5.4. *With notations set in the statement of Theorem 5.3, set $\mathcal{H}(s)=\mathcal{H}(s,s_0)$. Denote the left and right derivative of a function λ at $s\in\mathbb{R}$ by $\lambda'_-(s)$ and $\lambda'_+(s)$ respectively. In the context of situation (A) or (B) stated in Theorem 5.3,*

1. *On the interval (s_0,∞), the derivative of $\mathcal{H}$ exists except for countably many points. The left and right derivative of $\mathcal{H}$ exists everywhere. The second derivative of $\mathcal{H}$ exists almost everywhere, i.e. outside a set of Lebesgue measure zero.*
2. *The left and right derivatives of $\mathcal{H}$ are both decreasing in terms of s. We have $\mathcal{H}'_+(s)\leq\mathcal{H}'_-(s)$, and if $s_1<s_2$, $\mathcal{H}'_-(s_2)\leq\mathcal{H}'_+(s_1)$.*
3. *On the interval $(0,\infty)$, the derivative of h exists except for countably many points. The left and right derivative of h exists everywhere. The second derivative of h exists almost everywhere.*
4. *On (s_0,∞), $h'_+(s)=\mathcal{H}'_+(s)c(s)$, $h'_-(s)=\mathcal{H}'_-(s)c(s)$ exists, and $h'_+(s)\leq h'_-(s)$ for any $s\in(0,\infty)$.*

Proof. (1) and (2) follows from properties of convex functions and the convexity of $\mathcal{H}$ in Theorem 5.3, (2).

(3), (4): Recall

$$\mathcal{H}(s, s_0) = h_{M,I,J_\bullet}(s)/c(s) - h_{M,I,J_\bullet}(s_0)/c(s_0) + \int_{s_0}^{s} h_{M,I,J_\bullet}(t)c'(t)/c^2(t)dt.$$

Since in the context of (A) and (B) $h_{M,I,J_\bullet}$ is continuous on $(0,\infty)$, the part of $\mathcal{H}(s,s_0)$ given by the integral is always differentiable. So (3) follows from the analogous properties of $\mathcal{H}(s,s_0)$ in (1) by varying s_0. The formulas in (4) follow from a direct computation. That $h'_+(s) \leq h'_-(s)$ follows from these formulas and (2). □

Remark 5.5. Trivedi asks when the Hilbert-Kunz density function of a graded pair (R, J) is $\dim(R) - 2$ times continuously differentiable; see [Tri21, Question 1]. In general the Hilbert-Kunz density function need not be $\dim(R) - 2$ times continuously differentiable; see [Muk23, Example 8.3.2]. Our work shows that the Hilbert-Kunz density function is always differentiable outside a set of measure zero. Indeed, a convex function on an interval is twice differentiable outside a set of measure zero; see [NP06, Section 1.4]. Thus from Theorem 5.3, it follows that outside a set of measure zero the h function is twice differentiable. Now from Theorem 6.7, we conclude that the Hilbert-Kunz density function of a graded domain of dimension at least two is differentiable outside a set of measure zero.

Remark 5.6. The conclusions of Theorem 5.3 and Theorem 5.4 are deduced in the context of situation (A) or (B), because we prove existence and continuity of $h_{M,I,J_\bullet}$ in those two contexts. So even outside the context of (A) or (B) whenever there is an h-function continuous on $(0,\infty)$, we have a corresponding version of Theorem 5.3 and Theorem 5.4.

We return to the question of existence of $f_{M,I,J_\bullet}(s)$ at a given $s \in \mathbb{R}$. We make comparisons between the limsup and and liminf of the sequence defining $f_{M,I,J_\bullet}(s)$ and the corresponding $h'_+(s)$ and $h'_-(s)$.

Lemma 5.7. *With the notation set in Theorem 5.3, set*

$$D_{n,t} = f_{n,M,I,J_\bullet}(t/p^n) = h_{n,M,I,J_\bullet}((t+1)/p^n) - h_{n,M,I,J_\bullet}(t/p^n).$$

In the context of situation (A) or (B),

(1)
$$h'_+(s) = \lim_{m\to\infty}\lim_{n\to\infty}\frac{\sum\limits_{t=\lceil sp^m p^n\rceil}^{\lceil sp^m p^n\rceil + p^n - 1} D_{m+n,t}}{p^{m(d-1)}p^{nd}}.$$

(2)
$$h'_-(s) = \lim_{m\to\infty}\lim_{n\to\infty}\frac{\sum\limits_{t=\lceil sp^m p^n\rceil - p^n}^{\lceil sp^m p^n\rceil - 1} D_{m+n,t}}{p^{m(d-1)}p^{nd}}.$$

Proof. (1) Note

$$\begin{aligned}
&\sum_{t=\lceil sp^m p^n\rceil}^{\lceil sp^m p^n\rceil+p^n-1} D_{m+n,t}\\
&= \sum_{t=\lceil sp^m p^n\rceil}^{\lceil sp^m p^n\rceil+p^n-1} f_{m+n,M}(t/p^m p^n)\\
&= h_{m+n,M}(\lceil sp^m p^n\rceil/p^m p^n + 1/p^m) - h_{m+n,M}(\lceil sp^m p^n\rceil/p^m p^n)\ .
\end{aligned}$$

Since in the context of (A) or (B), the h-function exists, the right hand side of the desired equation in (1) is

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty}\frac{h_{m+n,M}(\lceil sp^m p^n\rceil/p^m p^n + \frac{1}{p^m}) - h_{m+n,M}(\lceil sp^m p^n\rceil/p^m p^n)}{p^{m(d-1)}p^n}\\
&= \lim_{m\to\infty}\frac{h_M(s+1/p^m)-h_M(s)}{1/p^m}\\
&= h'_+(s)\ .
\end{aligned}$$

(2) Note

$$\begin{aligned}
&\sum_{t=\lceil sp^m p^n\rceil-p^n}^{\lceil sp^m p^n\rceil-1} D_{m+n,t}\\
&= \sum_{t=\lceil sp^m p^n\rceil-p^n}^{\lceil sp^m p^n\rceil-1} f_{m+n,M}(t/p^m p^n)\\
&= h_{m+n,M}(\lceil sp^m p^n\rceil/p^m p^n) - h_{m+n,M}(\lceil sp^m p^n\rceil/p^m p^n - 1/p^m)
\end{aligned}$$

Thus the right hand side of the desired equation in (1) is

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty}\frac{h_{m+n,M}(\lceil sp^m p^n\rceil/p^m p^n) - h_{m+n,M}(\lceil sp^m p^n\rceil/p^m p^n - 1/p^m)}{p^{m(d-1)}p^n}\\
&= \lim_{m\to\infty}\frac{h_M(s)-h_M(s-1/p^m)}{1/p^m}\\
&= h'_-(s)\ .
\end{aligned}$$

□

Theorem 5.8. *With the same notation as in Theorem 5.3, in the context of situation (A) or (B),*

(1) *for any $s>0$,*

$$h'_+(s) \le \underline{\lim}_{n\to\infty} f_{n,M,I,J_\bullet}(s)/p^{n(d-1)} \le \overline{\lim}_{n\to\infty} f_{n,M,I,J_\bullet}(s)/p^{n(d-1)} \le h'_-(s),$$

where $\underline{\lim}$ and $\overline{\lim}$ denote liminf and limsup respectively.

(2) *At $s>0$, if h_M is differentiable, then $f_{M,I,J_\bullet}(s)$- the density function of $(M,I,J_\bullet)$ at s exists and is equal to $h'_{M,I,J_\bullet}(s)$. If $h_M(s)$ is a C^1-function, then $f_M(s)$ is continuous.*

(3) *There is a countable subset of $(0,\infty)$ outside which $f_{M,I,J_\bullet}(s)$ exists and is equal to $h'_{M,I,J_\bullet}(s)$.*

Proof. (1) In the proof, we also use the notation set in Lemma 5.7, (1). Set

$$\alpha_{\mu,t} = \binom{\mu+t-1}{\mu-1}.$$

Note $D_{n,t} = l((I^t + J_n)M/(I^{t+1} + J_n)M)$. For a fixed n, $D_{n,t}/\alpha_{\mu,t}$ is a decreasing function of t, thanks to Lemma 3.17. So for $\lceil sp^m p^n \rceil \leq t \leq \lceil sp^m p^n \rceil + p^n - 1$, $D_{m+n,t}/\alpha_{\mu,t} \leq D_{m+n,\lceil sp^m p^n\rceil}/\alpha_{\mu,\lceil sp^m p^n\rceil}$, so

$$\begin{aligned} D_{m+n,t} &\leq D_{m+n,\lceil sp^m p^n\rceil} \frac{\alpha_{\mu,t}}{\alpha_{\mu,\lceil sp^m p^n\rceil}} \\ &\leq D_{m+n,\lceil sp^m p^n\rceil} \frac{\alpha_{\mu,\lceil sp^m p^n\rceil + p^n}}{\alpha_{\mu,\lceil sp^m p^n\rceil}} \ . \end{aligned}$$

Also $\alpha_{\mu,t}$ is a polynomial of degree $\mu - 1$ in t, so

$$\begin{aligned} \lim_{m\to\infty}\lim_{n\to\infty} \frac{\alpha_{\mu,\lceil sp^m p^n\rceil + p^n}}{\alpha_{\mu,\lceil sp^m p^n\rceil}} &= \lim_{m\to\infty}\lim_{n\to\infty} \frac{(\lceil sp^m p^n\rceil + p^n)^{\mu-1}}{\lceil sp^m p^n\rceil^{\mu-1}} \\ &= \lim_{m\to\infty} \frac{(sp^m+1)^{\mu-1}}{(sp^m)^{\mu-1}} \\ &= 1. \end{aligned}$$

So

$$\begin{aligned} h'_+(s) &= \lim_{m\to\infty}\lim_{n\to\infty} \frac{\sum_{t=\lceil sp^m p^n\rceil}^{\lceil sp^m p^n\rceil + p^n - 1} D_{m+n,t}}{p^{m(d-1)}p^{nd}} \\ &\leq \underline{\lim}_{m\to\infty}\underline{\lim}_{n\to\infty} \frac{p^n D_{m+n,\lceil sp^m p^n\rceil}}{p^{m(d-1)}p^{nd}} \frac{\alpha_{\mu,\lceil sp^m p^n\rceil + p^n}}{\alpha_{\mu,\lceil sp^m p^n\rceil}} \\ &= \underline{\lim}_{m\to\infty}\underline{\lim}_{n\to\infty} \frac{p^n D_{m+n,\lceil sp^m p^n\rceil}}{p^{m(d-1)}p^{nd}} \\ &= \underline{\lim}_{m\to\infty}\underline{\lim}_{n\to\infty} \frac{D_{m+n,\lceil sp^m p^n\rceil}}{p^{m(d-1)}p^{n(d-1)}} \ . \end{aligned}$$

For a sequence of real numbers β_n and any m, $\underline{\lim}_{n\to\infty}\beta_{m+n} = \underline{\lim}_{n\to\infty}\beta_n$ is independent of m, so $\underline{\lim}_{m\to\infty}\underline{\lim}_{n\to\infty}\frac{D_{m+n,\lceil sp^m p^n\rceil}}{p^{m(d-1)}p^{n(d-1)}} = \underline{\lim}_{n\to\infty}\frac{D_{n,\lceil sp^n\rceil}}{p^{n(d-1)}}$. Therefore we have

$$h'_+(s) \leq \underline{\lim}_{n\to\infty}\frac{D_{n,\lceil sp^n\rceil}}{p^{n(d-1)}} = \underline{\lim}_{n\to\infty}\frac{f_n(s)}{p^{n(d-1)}}.$$

The proof of the last inequality is similar. First we have If $\lceil sp^m p^n\rceil - p^n \leq t \leq \lceil sp^m p^n\rceil - 1$, then $D_{m+n,t}/\alpha_{\mu,t} \geq D_{m+n,\lceil sp^m p^n\rceil}/\alpha_{\mu,\lceil sp^m p^n\rceil}$, so

$$\begin{aligned} D_{m+n,t} &\geq D_{m+n,\lceil sp^m p^n\rceil} \frac{\alpha_{\mu,t}}{\alpha_{\mu,\lceil sp^m p^n\rceil}} \\ &\geq D_{m+n,\lceil sp^m p^n\rceil} \frac{\alpha_{\mu,\lceil sp^m p^n\rceil - p^n}}{\alpha_{\mu,\lceil sp^m p^n\rceil}} \ . \end{aligned}$$

Also $\alpha_{\mu,t}$ is a polynomial of degree $\mu-1$ in t, so

$$\begin{aligned}\lim_{m\to\infty}\lim_{n\to\infty}\frac{\alpha_{\mu,\lceil sp^mp^n\rceil-p^n}}{\alpha_{\mu,\lceil sp^mp^n\rceil}} &= \lim_{m\to\infty}\lim_{n\to\infty}\frac{(\lceil sp^mp^n\rceil-p^n)^{\mu-1}}{\lceil sp^mp^n\rceil^{\mu-1}}\\ &= \lim_{m\to\infty}\frac{(sp^m+1)^{\mu-1}}{(sp^m)^{\mu-1}}\\ &= 1.\end{aligned}$$

So

$$\begin{aligned}h'_-(s) &= \lim_{m\to\infty}\lim_{n\to\infty}\frac{\sum_{t=\lceil sp^mp^n\rceil-p^n}^{\lceil sp^mp^n\rceil-1} D_{m+n,t}}{p^{m(d-1)}p^{nd}}\\ &\geq \overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty}\frac{p^nD_{m+n,\lceil sp^mp^n\rceil}}{p^{m(d-1)}p^{nd}}\frac{\alpha_{\mu,\lceil sp^mp^n\rceil-p^n}}{\alpha_{\mu,\lceil sp^mp^n\rceil}}\\ &= \overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty}\frac{p^nD_{m+n,\lceil sp^mp^n\rceil}}{p^{m(d-1)}p^{nd}}\\ &= \overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty}\frac{D_{m+n,\lceil sp^mp^n\rceil}}{p^{m(d-1)}p^{n(d-1)}}\ .\end{aligned}$$

For a sequence of real numbers β_n and any m, $\overline{\lim}_{n\to\infty}\beta_{m+n}=\overline{\lim}_{n\to\infty}\beta_n$ is independent of m, so $\overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty}\frac{D_{m+n,\lceil sp^mp^n\rceil}}{p^{m(d-1)}p^{n(d-1)}}=\overline{\lim}_{n\to\infty}\frac{D_{n,\lceil sp^n\rceil}}{p^{n(d-1)}}$. Therefore we have

$$h'_-(s)\geq\overline{\lim}_{n\to\infty}\frac{D_{n,\lceil sp^n\rceil}}{p^{n(d-1)}}=\overline{\lim}_{n\to\infty}\frac{f_n(s)}{p^{n(d-1)}}.$$

(2) If h_M is differentiable at s, $h'_+(s)=h'_-(s)$. Thus (1) implies that $f_{n,M}(s)/q^{d-1}$ exists and is equal to $h'(s)$, rest of (2) is clear.

(3) folows from Theorem 5.4, (3). □

Remark 5.9. We prove Theorem 5.8 in the context of situation (A) or (B) defined in Theorem 5.3- which is precisely the contexts where we prove existence of $h_{M,I,J_\bullet}$ in this article. Thus when $(R,\mathfrak{m})$ is a domain, $I,J_\bullet$ satisfy **Condition C**, we get a corresponding density function which is well-defined outside a countable subset of $(0,\infty)$. One particular special case, potentially important for its application to prime characteristic singularity theory, is when $J_\bullet$ is the ideal sequence that defines the F-signature of $(R,\mathfrak{m})$; see Example 3.10.

When $J_n=J^{[q]}$, Theorem 5.8 yields a Hilbert-Kunz density function of (I,J) well defined outside a countable subset of $(0,\infty)$.

The function $h_{M,I,J_\bullet}$ need not be continuous or differentiable at zero. In Theorem 8.11, we prove that $h_{R,I,J}$ is continuous at zero if and only if $\dim R-\dim R/I\geq 1$ and differentiable at zero if and only if $\dim R-\dim R/I\geq 2$.

Example 5.10. We point out that the h-function need not be differentiable on $(0,\infty)$. Our example of a non differentiable h-function comes from [BST13]. Fix a regular local domain $(R,\mathfrak{m})$ of dimension d and a non-zero $f\in R$. For $t\in\mathbb{R}$, [BST13] considers the function $t\to s(R,f^t)$: the F-signature of the pair (R,f^t) which is shown to be the same as

$$s(R,f^t)=\lim_{n\to\infty}\frac{1}{q^d}l(\frac{R}{\mathfrak{m}^{[p^n]}:f^{\lceil tp^n\rceil}}).$$

With $I=(f)$, $h_{R,I,\mathfrak{m}}(t)=1-s(R,f^t)$; see [BST13, section 4]. At $t=1$, the left hand derivative of h_I is the F-signature of R/f; see [BST13, Thm 4.6], while the right hand derivative is zero since $h(s) = 1$ for $s \geq 1$. So h is not differentiable at one if and only if the F-signature of R/f is non-zero, precisely when R/f is strongly F-regular. A concrete example comes from the strongly F-regular ring, $\mathbb{F}_p[[x, y, z]]/(x^2 + y^2 + z^2)$ with $p \geq 3$.

Example 5.11. We point out that the limit defining the density function at a particular $s \in \mathbb{R}$, i.e. of $f_{n,M,I,J}(s)/q^{\dim(M)-1}$ may not converge. For example, when $I = 0$, $M = R$, then $f_{n,M,I,J}(0) = l(R/J^{[q]})$; thus $f_{n,M,I,J}/q^{\dim R} = e_{HK}(J, R)$ is a nonzero real number, so $f_{n,M,I,J}/q^{\dim R-1}$ goes to infinity. This example implies that some assumption is necessary to guarantee the existence of the density function at every point.

Example 5.12. In the definition of the density function if we replace $\lceil sq \rceil$ by $\lfloor sq \rfloor$, then we have more examples where the density function does not exist. We recall Otha's example mentioned in [Kos17, sec 3] which produces such instances. Let R be the power series ring $k[[x_1, \dots, x_{d+1}]]$, $\alpha_1 \leq \dots \leq \alpha_{d+1}$ be a sequence of positive integers, $I = (x_1^{\alpha_1} \dots x_{d+1}^{\alpha_{d+1}})$ be a monomial principal ideal, $J = (x_1, \dots, x_{d+1})$ be the maximal ideal of R. Assume moreover that $\alpha_d < \alpha_{d+1}$, α_{d+1} does not divide p,and ϵ_n is the residue of p^n modulo α_{d+1}. Let $\tilde{f}$ be the density function defined using $\lfloor sq \rfloor$, then $\lim_{n\to\infty} \tilde{f}_{n,R,I,J}/(p^{nd}\epsilon_n)$ exists and is nonzero, so $\lim_{n\to\infty} \tilde{f}_{n,R,I,J}/p^{nd}$ exists if and only if ϵ_n is a constant sequence, and this is false in general. In general, ϵ_n is a periodic function and its period is the order of $p + \alpha_{n+1}\mathbb{Z}$ in the multiplicative group $(\mathbb{Z}/\alpha_{n+1}\mathbb{Z})^*$.

Example 5.13. We give an example, where the density function exists everywhere although the h-function is not differentiable everywhere. Note that the resulting density function is not continuous in this case; compare with Theorem 6.4. Let $M = R = k[[x]]$ be the power seires ring, $I = J = (x)$. Then $h_n(s) = l(R/I^{\lceil sq \rceil} + J^{[q]}) = min\{\lceil sq \rceil, q\}$. By simple calculation we get $f_n(s) = 1$ when $-1/q < s \leq 1 - 1/q$ and is 0 otherwise. So $f(s) = 1$ when $0 \leq s < 1$ and $f(s) = 0$ otherwise.

Here f_n converges pointwise but not uniformly. Outside an arbitrary neighborhood of 0 and 1 then f_n converges uniformly.

On the other hand, $h(s)$ is 0 when $s \leq 0$, s when $0 \leq s \leq 1$, 1 when $s \geq 1$, and is continuous. We have $f(s) = h'(s)$ when $s \neq 0, 1$; when $s = 0, 1$ $h'(s)$ does not exist and $f(s) = h'_+(s)$. This leads us to guessing that whenever the density function exists at s, it coincides with the right hand derivative $h'_+(s)$.

Remark 5.14. Assume $J_\bullet$ is big and $h_{M,I,J_\bullet}$ is differentiable everywhere. Since $h_{M,I,J_\bullet}$ is eventually constant (Lemma 3.8), the resulting density function $f_{M,I,J_\bullet} = h'_{M,I,J_\bullet}$ is supported on some compact interval $[0, b]$. So the density function has to increase and decrease on $[0, b]$. By Theorem 5.4, $f_{M,I,J_\bullet} = h'(s) = \mathcal{H}'(s)s^{\mu-1}/(\mu - 1)!$, where $\mathcal{H}'$ is decreasing since $\mathcal{H}$ is convex; so this gives a natural way to represent $f_{M,I,J_\bullet}$ as a product of a decreasing and an explicit increasing function, namely $c(s)$. This may help analyzing the monotonicity of the density function.

6. Relation among h, density, Frobenius-Poincaré functions

In Section 4 we developed a notion of Frobenius-Poincaré function in the local setting. Work of Section 5 gives a notion of Hilbert-Kunz density function in the local setting, at least outside a countable subset of $(0, \infty)$. When $(R, \mathfrak{m})$ is graded, we compare these local notions defined using the $\mathfrak{m}$-adic filtration with the classical notion of Frobenius-Poincaré function and Hilbert-Kunz density function defined (see Section 2) using the graded structure of the underlying objects.

Lemma 6.1. *Let $(R, \mathfrak{m})$ be a standard graded ring, M be a finitely generated $\mathbb{Z}$-graded module of dimension d, J be a homogeneous ideal of finite colength. Set*

$$g_{n,M,J,d-1}(s) = \frac{1}{q^{d-1}} l(\frac{M}{J^{[q]}M})_{\lceil sq \rceil},\ g_{n,M,J}(s) = l(\frac{M}{J^{[q]}M})_{\lceil sq \rceil}\ .$$

(1) *When M is generated in degree zero, for any graded submodule $N \subseteq M$ $(M/N)_j = \mathfrak{m}^j(M/N)/\mathfrak{m}^{j+1}(M/N)$.*
(2) *When M is generated in degree zero, $g_{n,M,J}(s) = l(\frac{M}{J^{[q]}M})_{\lceil sq \rceil} = f_{n,M,\mathfrak{m},J}(s)$.*

Proof. Let N be any submodule of M, then M/N is also generated in degree 0, so $(M/N)_{\geq j} = \mathfrak{m}^j(M/N)$ and $(M/N)_j = \mathfrak{m}^j(M/N)/\mathfrak{m}^{j+1}(M/N)$ for any j. This implies $g_{n,M,J}(s) = f_{n,M,\mathfrak{m},J}(s)$. □

Lemma 6.2. *We define an equivalence relation $\sim$ on graded modules over a standard graded ring R of positive dimension over a field: we say $M \sim N$ when there is a homogeneous map $\phi : M \to N$ such that $\dim Ker\phi, \dim Coker\phi \leq \dim R - 1$, and let $\sim$ also denote the minimal equivalence relation generated by such relations. Then M is equivalent to some module generated in degree 0.*

Proof. We can choose an element $c \in R_1$ such that $\dim R/cR \leq \dim R$. First, we find a sufficient large $n > 0$ such that M is generated in degree at most n. Then we truncate at degree n to get $M_{\geq n} := \oplus_{j=n}^{\infty} M_j$, which is generated in degree n. The module $M/M_{\geq n}$ is Artinian. The inclusion $M_{\geq n} \hookrightarrow M$ shows $M_{\geq n} \sim M$. The map $M_{\geq n} \to M_{\geq n}[n]$ given by multiplication by c^n has its kernel and cokernel annihilated by c^n. So the kernel and cokernel have dimension less than $\dim R$. Thus $M \sim M_{\geq n} \sim M_{\geq n}[n]$. Since $M_{\geq n}[n]$ is generated in degree zero, we are done. □

The next result follows directly from the lemma above and Proposition 3.32.

Lemma 6.3. *Let $(R, \mathfrak{m})$ be standard graded, M be a finitely generated $\mathbb{Z}$-graded R-module, I, $J_\bullet$ be homogeneous; assume that the corresponding objects obtained by localizing at $\mathfrak{m}$ satisfy condition (A) or (B) stated in Theorem 5.3. Then there is a finitely generated $\mathbb{N}$-graded R-module M' generated in degree zero such that, $h_{M,I,J_\bullet} = h_{M',I,J_\bullet}$.*

In the context of (A) or (B) stated in Theorem 5.3 there is an h-function and an associated density function defined outside a countable subset of $(0, \infty)$. Although the limit defining the density function may not exist at every point of $(0, \infty)$, we can define the integral of f on any bounded measurable subset Σ of $[0, \infty)$ by integrating the class in $L^1(\Sigma)$ represented by the density function. Fix the maximal subset Λ of $[0, \infty)$ where the density function $f_{M,I,J_\bullet}$ exists. The continuity of f_M at $s \in \Lambda$ refers to the notion of continuity coming from the subspace topology on the domain Λ inherited from $\mathbb{R}$. With this understanding, we have the following theorem.

Theorem 6.4. *Let $(R, \mathfrak{m})$, I, $J_\bullet$, M be as in Theorem 5.3. Then in the context of situation (A) or (B) as stated in Theorem 5.3, we have for any $s > 0$,*

$$h_{M,I,J_\bullet}(s) - \lim_{s_0 \to 0^+} h_{M,I,J_\bullet}(s_0) = \int_0^s f_{M,I,J_\bullet}(t)dt.$$

Moreover if the density $f_{M,I,J_\bullet}$ exists and is continuous at $s > 0$, then $h_{M,I,J_\bullet}$ is differentiable at s and $f_M(s) = h'_M(s)$.

Proof. Given $s > 0$, choose $[a, b] \subseteq \mathbb{R}_{>0}$ containing s. For a fixed s_0 in $[a, b]$ and $s > s_0$, we have

$$h_n(s) - h_n(s_0) = \sum_{j=\lceil s_0 q \rceil}^{\lceil sq \rceil - 1} f_n(\frac{j}{q}) \ .$$

Thus

$$\frac{1}{q^d} h_n(s) - \frac{1}{q^d} h_n(s_0) = \int_{s_0 - \frac{1}{q}}^{s - \frac{1}{q}} \frac{f_n(t)}{q^{d-1}} dt$$

By Theorem 3.19, we can choose a constant C such that for any $n \in \mathbb{N}$ and $t \in [a, b]$.

$$\frac{1}{q^{d-1}} f_n(t) \leq C.$$

Thus taking limit as n approaches infinity and using dominated convergence, we get

$$h_{M,I,J_\bullet}(s) - h_{M,I,J_\bullet}(s_0) = \int_{s_0}^{s} f_{M,I,J_\bullet}(t) dt.$$

Taking limit as $s_0 \to 0+$ we get the conclusion involving integrals. Note that $\lim_{s_0 \to 0+}$ exists as h is increasing.

Whenever $f_M(t)$ exists at s and is continuous at s, the differentiability of h_M at s and that $h'_M(s) = f_M(s)$ follows from the second fundamental theorem of Calculus. □

Proposition 6.5. *Continue with the same notation as in Lemma 6.1 but M not necessarily generated in degree zero. Set*

$$\tilde{g}_{n,M,J,d-1}(s) = l(M/J^{[q]}M)_{\lfloor sq \rfloor}/q^{d-1}.$$

If additionally $d = dim(M) \geq 2$, the two limits below exist for all $s \in \mathbb{R}$:

$$\tilde{g}_{M,J}(s) = \lim_{n \to \infty} \tilde{g}_{n,M,J,d-1}(s),\ g_{M,J}(s) = \lim_{n \to \infty} g_{n,M,J,d-1}(s).$$

Moreover $\tilde{g}_{M,J}(s) = g_{M,J}(s)$.

Proof. By [Tri18], $\tilde{g}_{n,M,J,d-1}(s)$ converges for all $s \in \mathbb{R}$. For $s \in \mathbb{Z}[1/p]$, $g_{n,M,J,d-1}(s) = \tilde{g}_{n,M,J,d-1}(s)$ for q large; so we conclude convergence of $g_{n,M,J,d-1}(s)$. When s is not in $\mathbb{Z}[1/p]$,

$$g_{n,M,J,d-1}(s) = \tilde{g}_{n,M,J,d-1}(s + \frac{1}{q}).$$

Now for $d \geq 2$, the uniform convergence of the sequence of functions $\tilde{g}_{n,M,J,d-1}$ and continuity of $\tilde{g}_{M,J}$ imply that the sequence $\tilde{g}_{n,M,J,d-1}(s + \frac{1}{q})$ converges to $\tilde{g}_{M,J}(s)$. □

Theorem 6.6. *Let $(R, \mathfrak{m})$ be standard graded, J be a homogeneous $\mathfrak{m}$-primary ideal, M an R-module of dimension $d \geq 2$. Then*

1. *$h_{M,\mathfrak{m},J}$ is differentiable on $\mathbb{R}$. The density function $f_{M,\mathfrak{m},J}(s)$ exists everywhere on $\mathbb{R}$ and is the same as $h'_{M,\mathfrak{m},J}(s)$.*
2. *Moreover $f_{M,\mathfrak{m},J}$ is the same as Trivedi's Hilbert-Kunz density function $\tilde{g}_{M,J}(s)$; see Section 2.*

Proof. (1) It follows from [Tay18, Lemma 3.3], that for $s \leq 1$, $h_M(s) = e(\mathfrak{m}, M)s^d/d!$. So h_M is differentiable at zero and the derivative is zero. A direct computation shows that the density function at zero exists and is zero. So we can restrict to $(0, \infty)$. Thanks to Theorem 5.8, (2), it is enough to show that h_M is differentiable on $(0, \infty)$. By using Lemma 6.3, we can assume that M is generated in degree zero. Thus by Lemma 6.1

$$f_{n,M,\mathfrak{m},J}(s) = g_{n,M,J}(s) := l([\frac{M}{J^{[p^n]}M}]_{\lceil sq \rceil}) \text{ for all } s \in \mathbb{R}.$$

As $d \geq 2$, by Proposition 6.5, $g_{n,M,J}(s)/q^{d-1}$ converges to Trivedi's density function $\tilde{g}_{M,J}(s)$ for all s. Since $\tilde{g}_{M,J}(s)$ is continuous, $f_{M,,J}(s)$ is also continuous. Now by Theorem 6.4, (2), $h_{M,I,J}$ is differentiable on $(0, \infty)$.

(2) Fix an M' which is generated in degree zero and equivalent to M in the sense of Lemma 6.2. Thanks to Lemma 6.3 and part (1)

$$h_M = h_{M'}\,, f_M = f_{M'}.$$

The associativity formula for Trivedi's density function implies (see [Tri18, Prop 2.14]), $\tilde{g}_{M,J} = \tilde{g}_{M',J}$. Since M' is generated in degree zero and has dimension at least two, by Lemma 6.1 and Proposition 6.5, $\tilde{g}_{M',J} = f_{M',\mathfrak{m},J}$. Putting together we conclude that $f_{M,\mathfrak{m},J} = \tilde{g}_{M,J}$. ☐

We further strengthen the above theorem by proving it for any homogeneous J which not necessarily has finite colength,

Theorem 6.7. *Let $(R, \mathfrak{m})$ be a standard graded, J be a homogeneous ideal, $s \in \mathbb{R}$, M be a finitely generated graded module of dimension d. Assume $d \geq 2$. Set $\tilde{g}_{n,M,J,d-1}(s) = l(M/J^{[q]}M)_{\lfloor sq \rfloor}/q^{d-1}$. Then:*

1. *The sequence $(\tilde{g}_{n,M,J,d-1}(s))_n$ converges uniformly on every compact subset of $\mathbb{R}$. The limiting function is continuous.*
2. *$h_{M,\mathfrak{m},J}$ is differentiable and*

$$h'_{M,\mathfrak{m},J}(s) = f_{M,\mathfrak{m},J}(s) = \lim_{n\to\infty} \tilde{g}_{n,M,J,d-1}(s).$$

Proof. (1) For a positive integer N, set $J' = J + \mathfrak{m}^{N+1}$. Then on $[0, N]$, $\tilde{g}_{n,M,J,d-1} = \tilde{g}_{n,M,J',d-1}$. Since J' is $\mathfrak{m}$-primary, by [Tri18], $\tilde{g}_{n,M,J',d-1}$ converges uniformly to a continuous function. Thus on $[0, N]$, $\tilde{g}_{n,M,J,d-1}$ converges uniformly to a continuous function.

(2) Fix a compact interval $[a, b] \subseteq \mathbb{R}$. By Theorem 3.13, (1), we can choose t_0 such that for all $t \geq t_0$, $h_{M,\mathfrak{m},J} = h_{M,\mathfrak{m},J+\mathfrak{m}^t}$ on $[a, b]$. Using the ideas from the argument in part(1), fix an integer $t \geq t_0$, ensure $\tilde{g}_{n,M,J,d-1} = \tilde{g}_{n,M,J+\mathfrak{m}^t,d-1}$ on $[a, b]$ for all n. By Theorem 6.6, $h_{M,\mathfrak{m},J+\mathfrak{m}^t}$ is differentiable on $\mathbb{R}$ with derivative $\tilde{g}_{M,J+\mathfrak{m}^t}$. Thus on (a, b), $h_{M,\mathfrak{m},J}$ is differentiable with derivative being the continuous function $\tilde{g}_{M,J}$. Since by Theorem 5.8 $h'_M = f_M$ on (a, b), we are done. ☐

We point out below that in the graded context the Frobenius-Poincaré function defined using the underlying grading and the maximal ideal adic filtration coincide.. Recall that by Ω, we denote the open lower half complex plane. Let $(R, \mathfrak{m})$ be standard graded, M is an $\mathbb{N}$-graded R-module, J be a homogeneous ideal. For $y \in \Omega$,

Proposition 6.8. *Let $(R, \mathfrak{m})$ be standard graded, M an $\mathbb{N}$-graded R-module of dimension d, J be a homogeneous ideal. Consider the sequence of functions on the open lower half plane*

$$G_{n,M,J}(y) = \sum_{j=0}^{\infty} l([\frac{M}{J^{[q]}M}]_j)e^{-iyj/q}$$

1. *$\frac{1}{q^d}G_{n,M,J}(y)$ defines a holomorphic function on Ω for every n.*
2. *Recall that $F_{M,\mathfrak{m},J}$ denotes the Frobenius-Poincaré function defined in Definition 4.5. The sequence*

$$\lim_{n\to\infty} \frac{1}{q^d}G_{n,M,J}(y)$$

converges to $F_{M,\mathfrak{m},J}(y)$.

(3) *When J is $\mathfrak{m}$-primary, $G_{n,M,J}(y)/q^d$ converges to $F_{M,\mathfrak{m},J}(y)$ on $\mathbb{C}$.*

Proof. Fix an $\mathbb{N}$-graded module M' generated in degree zero and equivalent to M in the sense of Lemma 6.2.

(3) Since J is $\mathfrak{m}$-primary, G_n is a sum of finitely many entire functions. So G_n is entire. Fix a compact subset K of $\mathbb{C}$. By [Muk23, Lemma 3.2.5], we can find a constant D such that

$$|\frac{1}{q^d}G_{n,M,J}(y) - \frac{1}{q^d}G_{n,M',J}(y)| \leq \frac{D}{q} \text{ for all } n \text{ and } y \in K.$$

Since M' is generated in degree zero, $F_{n,M',\mathfrak{m},J} = G_{n,M',J}$. Since $F_{n,M',\mathfrak{m},J}/q^d$ uniformly converges to $F_{M',\mathfrak{m},J}$ on K, the last inequality implies that $\frac{1}{q^d}G_{n,M,J}$ converges uniformly to $F_{M',\mathfrak{m},J}$ on K; see Theorem 4.3. Thanks to Lemma 6.3 and Theorem 4.3, $F_{M',\mathfrak{m},J} = F_{M,\mathfrak{m},J}$ on $\mathbb{C}$.

(1) There is a polynomial P of degree d with non-negative coefficients such that

$$l([\frac{M}{J^{[q]}M}]_j) \leq l(M_j) \leq P(j).$$

Fix a compact subset $K \subseteq \Omega$. Choose $\epsilon > 0$ such that $\Im y < -\epsilon$ for every $y \in K$. Since

$$\sum_{j=0}^{\infty} \frac{1}{q^d}|P(j)|e^{-j\epsilon/q}$$

is convergent, we conclude that the sequence of holomorphic functions

$$(\frac{1}{q^d}\sum_{j=0}^{N} l([\frac{M}{J^{[q]}M}]_j)e^{-iyj/q})_N$$

converges uniformly to $\frac{1}{q^d}G_{n,M,J}(y)$ on K. This proves the holomorphicity of $\frac{1}{q^d}G_{n,M,J}$ on Ω.

(2) When $d = 0$, the conclusion follows from a direct computation. Assume $d \geq 1$. Since

$$l([\frac{M}{J^{[q]}M}]_j) = l([\frac{M}{J^{[q]}M}]_{\leq j}) - l([\frac{M}{J^{[q]}M}]_{\leq j-1}),$$

a direct computation using the equation above shows that,

$$\sum_{j=0}^{\infty} l([\frac{M}{J^{[q]}M}]_j)e^{-iyj/p^n} = \sum_{j=0}^{\infty} l([\frac{M}{J^{[q]}M}]_{\leq j})e^{-iyj/p^n}(1 - e^{-iy/p^n}) \ . \tag{6.1}$$

Since

$$l(\frac{(\mathfrak{m}^j + J^{[q]})M}{(\mathfrak{m}^{j+1} + J^{[q]})M}) = l([\frac{M}{(\mathfrak{m}^{j+1} + J^{[q]})M}]) - l([\frac{M}{(\mathfrak{m}^j + J^{[q]})M}]),$$

a direct computation shows that,

$$\sum_{j=0}^{\infty} l(\frac{(\mathfrak{m}^j + J^{[q]})M}{(\mathfrak{m}^{j+1} + J^{[q]})M})e^{-iyj/p^n} = \sum_{j=0}^{\infty} l(\frac{M}{(\mathfrak{m}^{j+1} + J^{[q]})M})e^{-iyj/p^n}(1 - e^{-iy/p^n}) \tag{6.2}$$

Choose a such that as an R-module M is generated by homogeneous elements of degree at most a. Therefore

$$\mathfrak{m}^j M \subseteq M_{\geq j} \subseteq \mathfrak{m}^{j-a}M.$$

So,

$$\begin{aligned}
l(\frac{M}{(\mathfrak{m}^{j+1}+J^{[q]})M})-l([\frac{M}{J^{[q]}M}]_{\leq j}) &= l(\frac{M_{\geq j+1}+J^{[q]}M}{\mathfrak{m}^{j+1}M+J^{[q]}M}) \\
&\leq l(\frac{\mathfrak{m}^{j+1-a}M+J^{[q]}M}{\mathfrak{m}^{j+1}M+J^{[q]}M}) \\
&\leq l(\frac{\mathfrak{m}^{j+1-a}M}{\mathfrak{m}^{j+1}M}) \\
&\leq Cj^{d-1},
\end{aligned}$$

for some C, which is independent of q and j. Using Equation (6.1), Equation (6.2) and the comparison above, we get that for any $y \in \Omega$,

$$\begin{aligned}
|\frac{1}{q^d}G_{n,M,J}(y)-\frac{1}{q^d}F_{n,\mathfrak{m},J}(y)| &\leq \sum_{j=0}^{\infty} C\frac{1}{q}(\frac{j}{q})^{d-1}e^{-\Im yj/q}|1-e^{-iy/q}| \\
&= C|1-e^{-iy/q}|\int_0^\infty \lfloor s\rfloor^{d-1}e^{-\Im y\lfloor s\rfloor}ds \\
&\leq C|1-e^{-iy/q}|\int_0^\infty s^{d-1}e^{-\Im y(s-1)}ds.
\end{aligned}$$

Since $\Im y < 0$ for $y \in \Omega$, the last integral is convergent. It follows from the last chain of inequalities that on a compact subset of Ω,

$$|\frac{1}{q^d}G_{n,M,J}(y)-\frac{1}{q^d}F_{n,\mathfrak{m},J}(y)|$$

uniformly converges to zero. This finishes the proof of (2). □

7. Arithmetic properties

In this section, we record some arithmetic properties of the function we have constructed in the previous sections.

7.1. $\mathfrak{m}$-adic continuity.

We have proven that the h-function is continuous with respect to the $\mathfrak{m}$-adic topology on the set of ideals in R.

Theorem 7.1. *Let $t \in \mathbb{N}$, I_t, J_t be two sequences of ideals such that $I_t + J_t \subset \mathfrak{m}^t$. Then for any s, $\lim_{t\to\infty} h_{M,I+I_t,J+J_t}(s) = h_{M,I,J}(s)$. This convergence is uniform with respect to s on any compact set in $(0,\infty)$.*

Proof. If $s \neq 0$ then both sides are 0, so there is nothing to prove. Fix $0 < s_1 < s_2 < \infty$ and it suffices to prove the uniform convergence on $[s_1, s_2]$, this is true by Theorem 3.13 and Theorem 3.20. □

The Frobenius-Poincaré function also satisfies a similar property:

Proposition 7.2. *Let $t \in \mathbb{N}$, I_t, J_t be two sequences of ideals such that $I_t + J_t \subset \mathfrak{m}^t$. Then for any $y \in \Omega$: the open lower half complex plane, $\lim_{t\to\infty} F_{M,I+I_t,J+J_t}(y) = F_{M,I,J}(y)$. If J is $\mathfrak{m}$-primary, then the above holds for $y \in \mathbb{C}$. In either case, the convergence is uniform on a compact subset of Ω or $\mathbb{C}$.*

Proof. Fix a compact subset K of Ω. Choose $\epsilon > 0$ such that $\Im y < -\epsilon$ for all $y \in K$. Recall from Theorem 3.16, that there is a polynomial $P \in \mathbb{R}[t]$ such that $h_{n,M,I,J}(s) \leq P(s)$ for all $s \in \mathbb{R}$ and all n; so $h_{M,I+I_t,J+J_t}(s) \leq P(s)$ for all s. Notice $|P(s)e^{-\epsilon s}|$ is integrable on $\mathbb{R}_{\geq 0}$ and the sequence $h_{M,I+I_t,J+J_t}$ converges to $h_{M,I,J}$; the convergence is uniform on every compact subset of $(0,\infty)$; see Theorem 3.13. Say the absolute values of elements of K is bounded above by D. Given $\delta > 0$, the observations above allows us to choose an interval $[a,b] \subseteq (0,\infty)$ and $t_0 \in \mathbb{N}$ such that,

(a) $2\int_0^a |P(s)|e^{-\epsilon s}ds + 2\int_b^\infty |P(s)|e^{-\epsilon s}ds \leq \frac{\delta}{2D}$.

(b) $|h_{M,I+I_t,J+J_t}(x) - h_{M,I,J}(x)| \leq \frac{\delta}{2D\int_a^b e^{-\epsilon s}ds}$ for all $t \geq t_0$ and all $s \in [a,b]$.

Therefore by using Theorem 4.3, for $y \in K$ and all $t \geq t_0$

$$\begin{aligned}|F_{M,I+I_t,J+J_t}(y) - F_{M,I,J}(y)| &\leq \int_0^\infty |y||h_{M,I+I_t,J+J_t}(s) - h_{M,I,J}(s)|e^{-\epsilon s}ds \\ &\leq D[2\int_0^a |P(s)|e^{-\epsilon s}ds + 2\int_b^\infty |P(s)|e^{-\epsilon s}ds \\ &+ \int_a^b |h_{M,I+I_t,J+J_t}(s) - h_{M,I,J}(s)|e^{-\epsilon s}ds] \\ &\leq \delta\ .\end{aligned}$$

This proves uniform convergence of $(F_{M,I+I_t,J+J_t}(y))_t$ to $F_{M,I,J}(y)$ on every compact subset of Ω. The assertion for $\mathfrak{m}$-primary J follows from a similar argument. ☐

7.2. **Basic properties.** Let R be a local ring, t be an indeterminate, I, J be $\mathfrak{m}$-primary ideals, M be a finitely generated R-module.

Theorem 7.3. *[Tay18, Proposition 2.6] Assume I, J are two $\mathfrak{m}$-primary ideals. Then*

(1) $\dim M < d$, *then* $h_{M,I,J,d}(s) = 0$.
(2) $h_{M,I,J}$ *is increasing.*
(3) $h_{M,I,J}(s) \leq e(I,M)s^d/d!$.
(4) $h_{M,I,J}(s) \leq e_{HK}(J,M)$.

Theorem 7.4. *The above (1) and (2) is still true if only $I+J$ is $\mathfrak{m}$-primary. (3) remains valid when I is $\mathfrak{m}$-primary and (4) remains valid when J is $\mathfrak{m}$-primary.*

Proof. By $\mathfrak{m}$-adic continuity $\lim_{t\to\infty} h_{M,I+\mathfrak{m}^t,J+\mathfrak{m}^t}(s) = h_{M,I,J}(s)$ and $I+\mathfrak{m}^t$, $J+\mathfrak{m}^t$ are $\mathfrak{m}$-primary. We have:

(1) $\dim M < d$, then $h_{M,I+\mathfrak{m}^t,J+\mathfrak{m}^t,d}(s) = 0$. Let $t \to \infty$, $h_{M,I,J,d}(s) = 0$.
(2) For $s_1 < s_2$, $h_{M,I+\mathfrak{m}^t,J+\mathfrak{m}^t}(s_1) \leq h_{M,I+\mathfrak{m}^t,J+\mathfrak{m}^t}(s_2)$. Let $t \to \infty$, $h_{M,I,J}(s_1) \leq h_{M,I,J}(s_2)$.
(3) $h_{M,I,J+\mathfrak{m}^t}(s) \leq e(I,M)s^d/d!$. Let $t \to \infty$, we have $h_{M,I,J}(s) \leq e(I,M)s^d/d!$.
(4) $h_{M,I+\mathfrak{m}^t,J}(s) \leq e_{HK}(J,M)$. Let $t \to \infty$, we have $h_{M,I,J}(s) \leq e_{HK}(J,M)$.

☐

Proposition 7.5. *[Additivity]Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of modules of dimension at most d. Let I, J be ideals such that $I+J$ is $\mathfrak{m}$-primary. Recall that the Kronecker delta notation $\delta_{a,b}$ represents zero if $a \neq b$ and 1 if $a = b$.*

(1) $\mathcal{F}_{M,I,J} = \delta_{dim(M),dim(M')}\mathcal{F}_{M',I,J} + \delta_{dim(M),dim(M'')}\mathcal{F}_{M'',I,J}$ *for* $\mathcal{F} = h, F$;
(2) $f_M(s) = \delta_{dim(M),dim(M')}f_{M'}(s) + \delta_{dim(M),dim(M'')}f_{M''}(s)$, *whenever* $h_{M,I,J}$, $\delta_{dim(M),dim(M')}h_{M',I,J}$, $\delta_{dim(M),dim(M'')}$, $h_{M'',I,J}$ *are all differentiable at* s.

Proof. (1)When $\mathcal{F} = h$, this is true by Proposition 3.32. Then Theorem 4.3 implies the statement for $\mathcal{F} = F_M$.
(2) follows from Theorem 5.8. □

Corollary 7.6 (Associativity formula)**.** The h-function, density function and Frobenius-Poincaré function satisfy the associativity formula. To be precise,

(1) let $\mathcal{F} \in \{h, F\}$, then
$$\mathcal{F}_{M,d}(s) = \sum_{P \in \mathrm{Spec}(R), \dim R/P = \dim R} \lambda_{R_P}(M_P)\mathcal{F}_{R/P}(s),$$
for all $s \in \mathbb{R}$.
(2) At a point s where $h_{R/P}$ is differentiable for all $P \in \mathrm{Assh}(R)$, the same associativity formula holds for the density function (i.e. $\mathcal{F} = f$) at s.

Theorem 7.7. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring of dimension d, M be a finitely generated module of dimension d, I, I', J, J' be R-ideals such that $I' \subset I, J' \subset J$, $I' + J'$ is $\mathfrak{m}$-primary. Then $h_{M,I',J'}(s) \geq h_{M,I,J}(s)$ and equality holds if $I \subset \bar{I}'$ and $J \subset J'^*$.*

Proof. The first part of (3) is clear. Both sides are additive on M, so by the associativity formula, we can replace M with R/P where $\dim R/P = d$. The containment hypotheses on the ideals also hold for their images in R/P for any prime ideal P. So we may assume $M = R$ and R is a domain. By definition of the integral closure and tight closure we can choose a nonzero $c \in R$ such that $cI^n \subset I'^n$ and $cJ^{[q]} \subset J'^{[q]}$, thus $I^{\lceil sq \rceil} + J^{[q]}/I'^{\lceil sq \rceil} + J'^{[q]}$ is annihilated by c. So
$$\begin{aligned} & l(\frac{I^{\lceil sq \rceil} + J^{[q]}}{I'^{\lceil sq \rceil} + J'^{[q]}}) \\ & \leq l(0 :_{\frac{R}{I'^{\lceil sq \rceil} + J'^{[q]}}} c) \\ = & l(\frac{R}{cR + I'^{\lceil sq \rceil} + J'^{[q]}}) \leq Cq^{d-1} \end{aligned}$$
The last equation is true because $\dim R/cR < \dim R$. This means
$$0 \leq h_{n,M,I',J'}(s) - h_{n,M,I,J}(s) \leq Cq^{d-1}.$$
Dividing by q^d and take the limit when $q \to \infty$, we get $h_{M,I',J'}(s) = h_{M,I,J}(s)$. □

Theorem 7.8. *Let $n_0 \in \mathbb{N}$, then*
$$h_{M,I^{n_0},J}(s) = h_{M,I,J}(sn_0), h_{M,I,J^{[p^{n_0}]}}(s) = p^{n_0 d} h_{M,I,J}(s/p^{n_0}).$$

Proof. If $s \leq 0$ then both sides of the equation are 0 and the equality holds. Now we assume $s > 0$. By definition $h_{n,M,I^{n_0},J}(s) = l(M/I^{n_0 \lceil sq \rceil} + J^{[q]}M)$. Since $\lceil sqn_0 \rceil \leq n_0 \lceil sq \rceil \leq \lceil sqn_0 \rceil + n_0 - 1$, $h_{n,M,I,J}(sn_0) \leq h_{n,M,I_0^n,J}(s) \leq h_{n,M,I,J}(sn_0 + (n_0 - 1)/q)$. We have $\lim_{n \to \infty}(h_{n,M,I,J}(sn_0 + (n_0 - 1)/q) - h_{n,M,I,J}(sn_0))/q^d = 0$ by Theorem 3.20. So
$$\lim_{n \to \infty} h_{n,M,I_0^n,J}(s)/q^d = \lim_{n \to \infty} h_{n,M,I,J}(sn_0)/q^d,$$
which means $h_{M,I_0^n,J}(s) = h_{M,I,J}(sn_0)$. We have $h_{n,M,I,J^{[p^{n_0}]}}(s) = l(M/I^{\lceil sq \rceil} + J^{[qp_0^n]}M) = l(M/I^{\lceil s/p^{n_0} \cdot qp^{n_0} \rceil} + J^{[qp_0^n]}M)$. So
$$\begin{aligned} & \lim_{n \to \infty} \frac{h_{n,M,I,J^{[p^{n_0}]}}(s)}{q^d} \\ = & p^{n_0 d} \lim_{n \to \infty} \frac{h_{n+n_0,M,I,J}(s/p^{n_0})}{q^d p^{n_0 d}} \\ & = p^{n_0 d} h_{M,I,J}(s/p^{n_0}). \end{aligned}$$

$\square$

7.3. **Integration and h-function.** Let R be a local ring of characteristic p, $R[[t]]$ be a power series ring with indeterminate t. Let M be a finitely generated R-module, I, J be two R-ideals such that $I+J$ is $\mathfrak{m}$-primary. Let $M[[t]] = M \otimes_R R[[t]]$. We want to express $h_{M[[t]],R[[t]],(I,t^\alpha),(J,t^\beta)}$ in terms of $h_{M,R,I,J}$.

Theorem 7.9. *(1)*$h_{M[[t]],R[[t]],(I,t^\alpha),(J,t^\beta)}(s) = \alpha \int_{s-\beta/\alpha}^{s} h_{M,R,I,J}(x)dx$
(2)$h_{M[[t]],R[[t]],(I,t^\alpha),J}(s) = \alpha \int_0^s h_{M,R,I,J}(x)dx$
(3)$h_{M[[t]],R[[t]],I,(J,t^\beta)}(s) = \beta h_{M,R,I,J}(s)$.

Proof. We will use the convention $I^s = R$ when $s \leq 0$. To prove the equality we may assume $s = s_0/q_0 \in \mathbb{Z}[1/p]$ because the functions on both sides are continuous when $s > 0$. Then for $q \geq q_0$, sq is an integer.

$$h_{n,M[[t]],R[[t]],(I,t^\alpha),(J,t^\beta)} = l(\frac{M[[t]]}{((I,t^\alpha)^{sq} + (J^{[q]}, t^{\beta q}))M[[t]]})$$

The above length is also equal to

$$l(\frac{M[[t]]}{(\sum_{0\leq j\leq sq} I^{sq-j}t^{\alpha j} + (J^{[q]}, t^{\beta q}))M[[t]]})$$

But by the convention, it is also

$$l(M[[t]]/\sum_{0\leq j\leq \infty} I^{sq-j}t^{\alpha j} + (J^{[q]}, t^{\beta q})M[[t]])$$

and because the existence of the $t^{\beta q}$-term, it is also equal to

$$l(M[[t]]/(\sum_{0\leq j\leq \lfloor \beta q/\alpha \rfloor} I^{sq-j}t^{\alpha j} + (J^{[q]}, t^{\beta q}))M[[t]])$$

Note that the module inside is nonzero only in t-degree at most $\beta q - 1$. So summing up over the lengths in different t-degrees, the above length is also equal to the following sum:

$$\sum_{0\leq x\leq \beta q-1} l(M/(J^{[q]} + I^{sq-\lfloor x/\alpha \rfloor})M)$$

Let $y = \lfloor x/\alpha \rfloor$. Up to adding a term of $O(q^d)$, it is equal to

$$\alpha \sum_{0\leq y\leq \lfloor \beta q/\alpha \rfloor} l(M/J^{[q]} + I^{sq-y}M)$$

which is exactly

$$\alpha \sum_{0\leq y\leq \lfloor \beta q/\alpha \rfloor} h_{n,M,I,J}(s - y/q)$$

$$= \alpha q \int_{s-\lfloor \beta q/\alpha \rfloor/q-1/q}^{s} h_{n,M,I,J}(x)dx$$

Now we divide by q^{d+1} and take the limit, then $O(q^d)$-term disappears, so the left is

$$= \alpha \int_{s-\beta/\alpha}^{s} h_{M,I,J}(x)dx.$$

Since the equation

$$h_{M[[t]],R[[t]],(I,t^\alpha),(J,t^\beta)} = \alpha \int_{s-\beta/\alpha}^{s} h_{M,R,I,J}(x)dx$$

is true on $\mathbb{Z}[1/p]$ and both sides are continuous with respect to s, they are equal on all of $\mathbb{R}$. The rest of the two equations can be obtained by taking limit as α or β goes to infinity and using the $\mathfrak{m}$-adic continuity proven in Theorem 3.13. □

7.4. **Ring extension.**

Proposition 7.10. *Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a local map such that $\mathfrak{m}S$ is $\mathfrak{n}$-primary and* $\dim R = \dim S$. *Then*

$$h_{M\otimes_R S,S,IS,JS}(s) \le l_S(S/\mathfrak{m}S)h_{M,R,I,J}(s).$$

The equality holds when S is flat over R.

Proof. For any $\mathfrak{m}$-primary ideal $\mathfrak{a}$, we have that $l_S(M\otimes_R S/(\mathfrak{a}S)M\otimes_R S) \le l_R(M/\mathfrak{a}M)l_S(S/\mathfrak{m}S)$. This means $h_{n,M\otimes_R S,S,IS,JS}(s) \le l(S/\mathfrak{m})h_{n,M,R,I,J}(s)$. All these equalities will hold if S is flat over R. □

8. Head and Tail of the h-function

In this section, we discuss the behaviour of $h(s)$ near zero and s large enough. The regions near zero and away from zero where the h-function often shows interesting behaviour are marked by two other already known invariants, namely F-limbus and F-threshold. F-threshold is a well-known numerical invariant in characteristic p which compares the ordinary power and Frobenius power; it was defined as a limsup in [Hun+08a] and [MTW04], and is shown to be a limit in [DNP18]. The F-limbus is less known, which is defined in [Tay18].

Definition 8.1. Let R be a ring of characteristic $p > 0$ which is not necessarily local, and let I, J be ideals of R. Define

$$c_I^J(n) = \sup\{t \in \mathbb{N} : I^t \not\subseteq J^{[p^n]}\}$$
$$c^J(I) = \lim_{n\to\infty} \frac{\sup\{t \in \mathbb{N} : I^t \not\subseteq J^{[p^n]}\}}{p^n}$$
$$b_I^J(n) = \inf\{t \in \mathbb{N} : J^{[p^n]} \not\subseteq I^t\}$$
$$b^J(I) = \lim_{n\to\infty} \frac{\inf\{t \in \mathbb{N} : J^{[p^n]} \not\subseteq I^t\}}{p^n}$$

The number $c^J(I)$ is called the F-threshold of I with respect to J and the number $b^J(I)$ is called the F-limbus of I with respect to J. The following properties are well known, For example, see [Tay18, Lemma 3.2].

Lemma 8.2. *Let R be a ring of characteristic $p > 0$, and let I, J be proper ideals of R.*

(1) *For any I, J, any limit above either exists or goes to infinity.*
(2) *If I is contained in the Jacobson radical of R, $I \not\subseteq nil(R)$, then $b^J(I) \le c^J(I)$.*
(3) *If $I \not\subseteq \sqrt{J}$ then $c^J(I) = \infty$.*
(4) *If $I \subset \sqrt{J}$ then $0 \le c^J(I) < \infty$.*
(5) *If $J \not\subseteq \sqrt{I}$ then $b^J(I) = 0$.*
(6) *If $J \subset \sqrt{I}$ then $0 < b^J(I) \le \infty$.*
(7) *If $I \subset Rad(R)$, $I \not\subseteq nil(R)$, $I \subset \sqrt{J}$, $J \subset \sqrt{I}$, then $0 < b^J(I) \le c^J(I) < \infty$.*

Lemma 8.3. *Let $(R,\mathfrak{m})$ be a local ring of dimension d and characteristic p, let I, J be two proper ideals of R, and let M be a finitely generated R-module.*

(1) *If I is $\mathfrak{m}$-primary, then $b^J(I) > 0$ and for $s \le b^J(I)$, $h_M(s) = \frac{s^d}{d!}e(I,M)$.*
(2) *If J is $\mathfrak{m}$-primary, then $c^J(I) < \infty$ and for $s \ge c^J(I)$, $h_M(s) = e_{HK}(J,M)$.*

Proof. The above Lemma is a generalization of Lemma 3.3 of [Tay18]. The proof is identically the same since it only uses the containment relation, which does not depend on whether I, J are $\mathfrak{m}$-primary or not. If I is $\mathfrak{m}$-primary then $J \subset \sqrt{I}$, so $b^J(I) > 0$; if J is $\mathfrak{m}$-primary then $I \subset \sqrt{J}$, so $c^J(I) < \infty$. □

8.1. **Tail of the h-function: F-threshold, minimal stable point and maximal support.** Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$, I, J are R-ideals. Assume J is $\mathfrak{m}$-primary. By Lemma 8.3, (2), when J is $\mathfrak{m}$-primary, the h-function becomes a constant $e_{HK}(J, M)$ when $s \gg 0$. Since $h(s)$ is increasing, $h_M(s) \leq e_{HK}(J, M)$ for any s. The h-function is also an increasing function, so there is a minimal point after which $h_{M,I,J}(s)$ becomes constant. Define

$$\alpha_{M,I,J} = \sup\{s | h_{M,I,J}(s) \neq e_{HK}(J, M)\} = \min\{s | h_{M,I,J}(s) = e_{HK}(J, M)\}.$$

We relate $\alpha_{R,I,J}$ to other seemingly unrelated invariants of (I, J).

Definition 8.4. Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$, I, J be two R-ideal, $I \subset \sqrt{J}$. Let

$$r_I^J(n) = \max\{t \in \mathbb{N} | I^t \not\subseteq (J^{[p^n]})^*\},$$

where $(J^{[p^n]})^*$ denotes the tight closure of $J^{[p^n]}$; see Definition 2.5.

$$r^J(I)^+ = \overline{\lim}_{n\to\infty} \frac{r_I^J(n)}{p^n}.$$

$$r^J(I)^- = \underline{\lim}_{n\to\infty} \frac{r_I^J(n)}{p^n}.$$

Under mild hypothesis, in Theorem 8.6, we show that $r^J(I)^+ = r^J(I)^- = \alpha_{R,I,J}$.

Lemma 8.5. *Let $(R, \mathfrak{m}, k)$ be a reduced d-dimensional local ring of characteristic $p > 0$, I, J be two R-ideals. Then $e_{HK}(J, R) = \lim_{n\to\infty} l(R/(J^{[q]})^*)/q^d$.*

Proof. It suffices to show $\lim_{n\to\infty} l((J^{[q]})^*/J^{[q]})/q^d = 0$. By assumption R is reduced, F-finite. So there is a test element $c \in R$, which is in particular not contained in any minimal prime of R; see [HH90, sec 6]. Since $c(J^{[q]})^* \subseteq J^{[q]}$ for all n, we have $l((J^{[q]})^*/J^{[q]}) \leq l(0_{R/J^{[q]}} : c) = l(R/cR + J^{[q]}) \leq Cq^{d-1}$ for some constant C, so $\lim_{n\to\infty} l((J^{[q]})^*/J^{[q]})/q^d = 0$. □

Theorem 8.6. *Let $(R, \mathfrak{m}, k)$ be a reduced formally equidimensional ring*[4] *of characteristic $p > 0$, I be an R-ideal, J be an $\mathfrak{m}$-primary R-ideal. Then $r^J(I)^+ = r^J(I)^- = \alpha_{R,I,J}$. In particular, $r^J(I) = \lim_{n\to\infty} r^J(I)(n)/p^n$ exists.*

Proof. Obviously $r^J(I)^+ \geq r^J(I)^-$, so it suffices to prove $r^J(I)^+ \leq \alpha_{R,I,J} \leq r^J(I)^-$. Since $\mathbb{Z}[1/p]$ is dense in $\mathbb{R}$, it suffices to prove:

(1) For $x \in \mathbb{Z}[1/p]$, if $x > r^J(I)^-$, then $x \geq \alpha_{R,I,J}$;
(2) For $x \in \mathbb{Z}[1/p]$, if $x < r^J(I)^+$, then $x \leq \alpha_{R,I,J}$.

(1): If $x > r^J(I)^-$, then there is an infinite sequence n_i, such that $xp^{n_i} > r^J(I)(n_i)$ and xp^{n_i} is an integer for all i. By definition of r_n, $I^{xp^{n_i}} \subset (J^{[p^{n_i}]})^*$. So

$$h_{R,I,J}(x) = \lim_{i\to\infty} l(R/I^{\lceil sp^{n_i}\rceil} + (J^{[p^{n_i}]})^*)/q^d = \lim_{i\to\infty} l(R/(J^{[p^{n_i}]})^*)/q^d = e_{HK}(J, R).$$

So $x \geq \alpha_{R,I,J}$.

(2): If $x < r^J(I)^+$, then there is a integer n_0, such that $xp^{n_0} \leq r^J(I)(n_0)$ and xp^{n_0} is an integer. Let $q_0 = p^{n_0}$. By definition of $r^J(I)(n)$, $I^{xq_0} \not\subseteq (J^{[q_0]})^*$. Choose $f \in I^{xq_0} \backslash (J^{[q_0]})^*$.

[4] i.e. the minimal primes of $\hat{R}$ have the same dimension

Let $\tilde{J} = J^{[q_0]} + fR$; then $e_{HK}(\tilde{J}, R) < e_{HK}(J^{[q_0]}, R)$; see [Hun13, Thm 5.5], [HH90, Thm 8.17]. Now fix an $s < xq_0$. Then for any $q = p^n$, $sq < xqq_0$. Since $f \in I^{xq_0}$. So $f^q \in I^{xqq_0} \subseteq I^{\lceil sq \rceil}$. So

$$I^{\lceil sq \rceil} + (J^{[q_0]} + fR)^{[q]} = I^{\lceil sq \rceil} + (J^{[q_0]})^{[q]}.$$

This means $h_{R,I,\tilde{J}}(s) = h_{R,I,J^{[q_0]}}(s)$. So for $s < xq_0$, $h_{R,I,J^{[q_0]}}(s) = h_{R,I,\tilde{J}}(s) \leq e_{HK}(\tilde{J}, R) < e_{HK}(J^{[q_0]}, R)$. This means $\alpha_{R,I,J^{[q_0]}} \geq xq_0$. By Theorem 7.8, $h_{R,I,J^{[q_0]}}(s) = q_0^d h_{R,I,J}(s/q_0)$, $\alpha_{R,I,J} = \frac{\alpha_{R,I,J^{[q_0]}}}{q_0} \geq x$. □

Since $h_M(s)$ is the integration of $f_M(s)$, we see the minimal stable point of h_M is the maximal support of f_M. Precisely,

Corollary 8.7. Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$, I be an R-ideal, J be an $\mathfrak{m}$-primary R-ideal. Then $\alpha_{R,I,J} = sup\{s | f_{R,I,J}(s) \text{ exists and is nonzero}\}$. Moreover for $s > \alpha_{R,I,J}$, $f_{R,I,J}(s)$ is zero.

Proof. For $s > \alpha_{R,I,J}$, $h_{I,J}(s)$ is constant. So by Theorem 5.8, $f_{I,J}$ exists and is zero. Since $h_{I,J}$ is the integral of the density function (Theorem 6.4) and h is a non-constant increasing function on $(0, \alpha_{R,I,J})$ for any $0 < a < \alpha_{R,I,J}$, $f_{I,J}$ has to be non-zero on a set of non-zero measure. □

Remark 8.8. Recall from Theorem 6.7 that for standard graded (R, m) of Krull dimension at least two and a finite colength homogeneous ideal J, Trivedi's density function $\tilde{g}_{R,J}$ coincides with $f_{R,\mathfrak{m},J}$ and both are continuous. So Theorem 8.6 gives a precise description of the support of $\tilde{g}_{R,J}$. Thus Theorem 8.6 and the theorem below extends [TW22, Thm 4.9], where $\alpha_{R,J}$ is shown to coincide with the F-threshold $c^J(\mathfrak{m})$ under suitable hypothesis.

Theorem 8.9. *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$, I be an R-ideal, J be an $\mathfrak{m}$-primary R-ideal. Then $c^J(I) = r^J(I)$ is true under either of the assumptions below:*

1. *There exists a sequence of positive numbers r'_n such that $I^{r'_n} \subset J^{[q]} : (J^{[q]})^*$ for infinitely many $q \gg 0$ and $\lim_n r'_n/p^n \to 0$.*
2. *There exists a constant n_0 such that $I^{n_0} \subset J^{[q]} : (J^{[q]})^*$ for infinitely many $q \gg 0$.*
3. *R is F-rational[5], i.e. the tight closure of every parameter ideal coincides with the ideal and J is a parameter ideal.*
4. *$I \subset \sqrt{\tau(R)}$, where $\tau(R) = \cap_{\mathfrak{a} \subset R} \mathfrak{a} : \mathfrak{a}^*$ is the test ideal of R. See [HH90, Definition 8.22, Proposition 8.23] for details on the test ideal.*
5. *(Theorem 4.9, [TW21])R is strongly F-regular on the punctured spectrum.*

Proof. (1) By definition $r_I^J(n) \leq c_I^J(n)$, and the condition implies $c_I^J(n) \leq r_I^J(n) + r_n$, so $\lim_n (c_I^J(n) - r_I^J(n))/p^n = 0$ and $c^J(I) = r^J(I)$.
(2) By (1) and the fact that $\lim_n n_0/n = 0$.
(3) If J is a parameter ideal, so is $J^{[q]}$. Since R is F-rational, $J^{[q]} : (J^{[q]})^* = R$ for any q, so $n_0 = 1$ satisfies the assumption of (2).
(4) There exist an n_0 such that $I^{n_0} \subset \tau(R) \subset \cap_q J^{[q]} : (J^{[q]})^*$, and this n_0 satisfies the assumption of (2).
(5) In this case $\tau(R)$ is either $\mathfrak{m}$-primary or is the unit ideal, so $I \subset \sqrt{\tau(R)}$ always holds.

□

[5]see [FW89], [Smi97]

8.2. **Head of the h-function: order of h_M at 0 and Hilbert-Kunz multiplicity of quotient rings.** So far we have proven continuity of the h-function on $\mathbb{R}_{>0}$; see Theorem 3.20, Theorem 3.31. In this section we determine when $h_{M,I,J}$ is continuous at $s=0$; see Theorem 8.13. In Theorem 8.11, we determine the order of vanishing of h-functions near the origin and show that the asymptotic behaviour of $h_{I,J}$ near the origin captures other numerical invariants of (R,I,J). A major intermediate step involved in proving Theorem 8.11 is Theorem 8.10, which boils down to proving commutation of the order of a double limit. We lay the groundwork for that.

Let $(R,\mathfrak{m},k)$ be a local ring of characteristic $p>0$, I,J be two R-ideals such that $I+J$ is $\mathfrak{m}$-primary. Let $d=\dim R$, $d'=\dim R/I$. For a positive integer s_0, consider the sequence of real numbers:

$$\Gamma_{s_0,m,n}=\frac{l(R/I^{s_0p^n}+J^{[p^np^m]})}{p^{nd}p^{md'}s_0^{d-d'}}.$$

$$\lim_{n\to\infty}\Gamma_{s_0,m,n}=\frac{h_R(s_0/p^m)}{(s_0/p^m)^{d-d'}}. \tag{8.1}$$

$$\begin{aligned}\lim_{m\to\infty}\Gamma_{s_0,m,n}&=\frac{e_{HK}(J^{[p^n]},R/I^{s_0p^n})}{p^{nd}s_0^{d-d'}}\\&=\frac{e_{HK}(J,R/I^{s_0p^n})}{(s_0p^n)^{d-d'}}\\&=\frac{1}{(s_0p^n)^{d-d'}}\sum_{P\in\operatorname{Assh}(R/I)}e_{HK}(J,R/P)l_{R_P}(R_P/I^{s_0p^n}R_P)\ .\end{aligned}$$

For $P\in\operatorname{Assh}(R/I)$, we have $ht(P)\le\dim R-\dim R/P=\dim R-\dim R/I=d-d'$. So

$$\begin{aligned}\lim_{n\to\infty}\lim_{m\to\infty}\Gamma_{s_0,m,n}&=\lim_{n\to\infty}\frac{1}{(s_0p^n)^{d-d'}}\sum_{P\in\operatorname{Assh}(R/I)}e_{HK}(J,R/P)l_{R_P}(R_P/I^{s_0p^n}R_P)\\&=\frac{1}{(d-d')!}\sum_{P\in\operatorname{Assh}(R/I)}e_{HK}(J,R/P)e(I,R_P)\ .\end{aligned}$$

Since R is F-finite domain and hence an excellent domain (see [Kun76]), for all $P\in\operatorname{Assh}(R/I)$, $ht(P)=d-d'$. So the above quantity is

$$\frac{1}{(d-d')!}\sum_{P\in\operatorname{Assh}(R/I)}e_{HK}(J,R/P)e(IR_P,R_P).$$

When R is a Cohen-Macaulay domain and I is part of a system of parameters, the above quantity recovers the Hilbert-Kunz multiplicity $e_{HK}(J,R/I)$ as,

$$\begin{aligned}&\sum_{P\in\operatorname{Assh}(R/I)}e_{HK}(J,R/P)e(IR_P,R_P)\\&=\sum_{P\in\operatorname{Assh}(R/I)}e_{HK}(J,R/P)l(R_P/IR_P)\\&=e_{HK}(J,R/I)\ .\end{aligned}$$

Theorem 8.10. *Assume R is a domain and $I \neq 0$ and J be such that $I+J$ is $\mathfrak{m}$-primary. Fix a positive integer s_0. Set dim$(R/I) = d'$. Then*

$$\lim_{m\to\infty} \frac{h(s_0/p^m)}{(s_0/p^m)^{d-d'}} = \frac{1}{(d-d')!} \sum_{P\in \mathrm{Assh}(R/I)} e_{HK}(J, R/P)e(I, R_P)\ .$$

Proof. We use the notation set above in this subsection. It follows from Equation (8.1) and above that we need to show

$$\lim_{m\to\infty}\lim_{n\to\infty} \Gamma_{s_0,m,n} = \lim_{n\to\infty}\lim_{m\to\infty} \Gamma_{s_0,m,n}.$$

We already see that $\lim_{n\to\infty}\Gamma_{s_0,m,n}$ and $\lim_{n\to\infty}\lim_{m\to\infty}\Gamma_{s_0,m,n}$ exists. We claim that that the sequence $n \to \Gamma_{s_0,m,n}$ is uniformly convergent in terms of m; then, by argument of analysis, we get $\lim_{m\to\infty}\lim_{n\to\infty}\Gamma_{s_0,m,n}$ exists, and is equal to $\lim_{n\to\infty}\lim_{m\to\infty}\Gamma_{s_0,m,n}$.

To this end, we prove that there exist a constant C such that $|\Gamma_{s_0,m,n+1}-\Gamma_{s_0,m,n}| \leq C/p^n$ for all m, which implies that $|\lim_{n\to\infty}\Gamma_{s_0,m,n} - \Gamma_{s_0,m,n}| \leq 2C/p^n$ for all m. We can prove it in two steps: we first prove there is a constant C_1 such that $\Gamma_{s_0,m,n+1} - \Gamma_{s_0,m,n} \leq C_1/p^n$, then we prove there is a constant C_2 such that $\Gamma_{s_0,m,n} - \Gamma_{s_0,m,n+1} \leq C_2/p^n$, then $C = max\{|C_1|, |C_2|\}$ satisfies the statement of the claim. Without loss of generality we assume $\frac{R}{m}$ is a perfect field; see Remark 3.15.

Choice of C_1: since $\dim R = d$, there is an exact sequence

$$0 \to R^{\oplus p^d} \to F_*R \to N \to 0$$

where N is an R-module with $\dim N < d$. Then we have

$$(R/I^{s_0p^n} + J^{[p^np^m]})^{\oplus p^d} \to F_*R/(I^{s_0p^n} + J^{[p^np^m]})F_*R \to N/I^{s_0p^n} + J^{[p^np^m]}N \to 0.$$

This means

$$l(\frac{R}{I^{s_0p^{n+1}} + J^{[p^{n+1}p^m]}}) \leq l(\frac{R}{I^{s_0p^n[p]} + J^{[p^{n+1}p^m]}}) \leq p^d l(\frac{R}{I^{s_0p^n} + J^{[p^np^m]}}) + l(\frac{N}{(I^{s_0p^n} + J^{[p^np^m]})N}).$$

So dividing $p^{(n+1)d}p^{md'}s_0^{d-d'}$, we get

$$\Gamma_{s_0,m,n+1} \leq \Gamma_{s_0,m,n} + l(N/I^{s_0p^n} + J^{[p^np^m]}N)/p^{(n+1)d}p^{md'}s_0^{d-d'}.$$

Now we claim that there is a constant $C_1 > 0$ that depends on N, I, J and s_0 but is independent of m, n such that $l(N/I^{s_0p^n} + J^{[p^np^m]}N)/p^{n(d-1)+d}p^{md'}s_0^{d-d'} \leq C_1$. We have

$$\begin{aligned} l(N/I^{s_0p^n} + J^{[p^np^m]}N) &\leq l(N/I^{s_0[p^n]} + J^{[p^np^m]}N) \\ &= l(F^n_*N/I^{s_0} + J^{[p^m]}F^n_*N) \\ &\leq \mu_R(F^n_*N)l(R/I^{s_0} + J^{[p^m]}). \end{aligned}$$

Since $\dim N \leq d-1$ and $\dim R/I = d'$, $\mu_R(F^n_*N)/p^{n(d-1)}$ and $l(R/I^{s_0} + J^{[p^m]})/p^{md'}$ are both bounded. And $p^{-d}s_0^{d-d'}$ is independent of m, n. This means there is a constant $C_1 > 0$ that depends on N, I, J and s_0 but is independent of m, n such that $l(N/I^{s_0p^n} + J^{[p^np^m]}N)/p^{n(d-1)+d}p^{md'}s_0^{d-d'} \leq C_1$. Thus we have

$$\Gamma_{s_0,m,n+1} \leq \Gamma_{s_0,m,n} + C_1/p^n.$$

Choice of C_2: since $\dim R = d$, there is an injection $F_*R \xrightarrow{\phi} R^{\oplus p^d}$ where $\dim Coker\phi < \dim R$. Let μ be the minimal number of generators of I. Choose $0 \neq c \in I$ and let $\psi = c^\mu\phi$. Since R is a domain, ψ is still an injection, and we have a short exact sequence

$$0 \to F_*R \xrightarrow{\psi} R^{\oplus p^d} \to N' \to 0$$

and we have $\dim N' < \dim R$.

$$F_*R/(I^{s_0p^n}+J^{[p^np^m]})F_*R \xrightarrow{\bar{\phi}} (R/I^{s_0p^n}+J^{[p^np^m]})^{\oplus p^d} \to N'/I^{s_0p^n}+J^{[p^np^m]}N' \to 0$$

We claim that $\bar{\phi}$ induces an R-linear map $\Phi : F_*(R/(I^{s_0p^{n+1}}+J^{[p^{n+1}p^m]})) \xrightarrow{\bar{\phi}} (R/I^{s_0p^n}+J^{[p^np^m]})^{\oplus p^d}$. It suffices to show $\psi(F_*(I^{s_0p^{n+1}}+J^{[p^{n+1}p^m]})) \in (I^{s_0p^n}+J^{[p^np^m]})^{\oplus p^d}$. We have $I^{s_0p^{n+1}} = I^{s_0p^np} \subset I^{(s_0p^n-\mu)[p]}$. So

$$\begin{aligned} &\psi(F_*(I^{s_0p^{n+1}}+J^{[p^{n+1}p^m]}))\\ \subset\, &\psi(F_*(I^{(s_0p^n-\mu)[p]}+J^{[p^{n+1}p^m]}))\\ \subset\, &I^{(s_0p^n-\mu)}+J^{[p^np^m]}\psi(F_*R)\\ \subset\, &c^\mu(I^{(s_0p^n-\mu)}+J^{[p^np^m]})\phi(F_*R)\\ \subset\, &I^{(s_0p^n)}+J^{[p^np^m]}\phi(F_*R)\\ \subset\, &(I^{(s_0p^n)}+J^{[p^np^m]})^{\oplus p^d}. \end{aligned}$$

This induces an exact sequence

$$F_*(R/(I^{s_0p^{n+1}}+J^{[p^{n+1}p^m]})) \to (R/I^{s_0p^n}+J^{[p^np^m]})^{\oplus p^d} \to N'/I^{s_0p^n}+J^{[p^np^m]}N' \to 0$$

Therefore,

$$p^d l(R/I^{s_0p^n}+J^{[p^np^m]}) \le l(R/I^{s_0p^{n+1}}+J^{[p^{n+1}p^m]}+l(N'/I^{s_0p^n}+J^{[p^np^m]}N')$$

So dividing $p^{(n+1)d}p^{md'}s_0^{d-d'}$, we get

$$\Gamma_{s_0,m,n+1} \le \Gamma_{s_0,m,n} + l(N'/I^{s_0p^n}+J^{[p^np^m]}N')/p^{(n+1)d}p^{md'}s_0^{d-d'}$$

Since $\dim N' < \dim R$, we can use the same proof in the previous step to show that there is a constant $C_2 > 0$ that depends on N', I, J and s_0 but independent of m, n such that $l(N'/I^{s_0p^n}+J^{[p^np^m]}N')/p^{n(d-1)+d}p^{md'}s_0^{d-d'} \le C_2$, so

$$\Gamma_{s_0,m,n} \le \Gamma_{s_0,m,n+1} + C_2/p^n.$$

□

Theorem 8.11. *Let $(R, \mathfrak{m}, k)$ be a local domain, I, J be two R-ideals, $I \ne 0$, $I+J$ is $\mathfrak{m}$-primary. Let $d = \dim R$, $d' = \dim R/I$. Then:*

(1) $\lim_{s\to 0+} h(s)/s^{d-d'} = \frac{1}{(d-d')!}\sum_{P\in \mathrm{Assh}(R/I)} e_{HK}(J, R/P)e(I, R_P)$.
(2) *The order of vanishing $h(s)$ at $s = 0$ is exactly $d - d'$.*
(3) *$h(s)$ is continuous at 0.*

Proof. (1) Let $\frac{1}{(d-d')!}\sum_{P\in \mathrm{Assh}(R/I), ht(P)=d-d'} e_{HK}(J, R/P)e(I, R_P) = c = c_{I,J}$, which is a constant that only depends on I, J. The last theorem implies for any fixed s_0,

$$\lim_{m\to\infty} h(s_0/p^m)/(s_0/p^m)^{d-d'} = c$$

Choose a sequence $\{s_i\}_i \subset (0, \infty)$ such that $\lim_{i\to\infty} s_i = 0$ and $\lim_{i\to\infty} h(s_i)/s_i^{d-d'}$ exists. Below we argue that $\lim_{i\to\infty} h(s_i)/s_i^{d-d'} = c$; then (1) follows. Fix any $n_0 \in \mathbb{N}$. There exists an integer α_i for each s_i such that $s_ip^{\alpha_i} \in (p^{n_0-1}, p^{n_0}]$. Since $h(s)$ is an increasing function,

$$\frac{h(\lfloor s_i q^{\alpha_i}\rfloor/q^{\alpha_i})}{((\lfloor s_i q^{\alpha_i}\rfloor+1)/q^{\alpha_i})^{d-d'}} \le \frac{h(s_i)}{s_i^{d-d'}} \le \frac{h(\lceil s_i q^{\alpha_i}\rceil/q^{\alpha_i})}{((\lceil s_i q^{\alpha_i}\rceil-1)/q^{\alpha_i})^{d-d'}}$$
$$\implies (\frac{\lfloor s_i q^{\alpha_i}\rfloor}{\lfloor s_i q^{\alpha_i}\rfloor+1})^{d-d'}\frac{h(\lfloor s_i q^{\alpha_i}\rfloor/q^{\alpha_i})}{(\lfloor s_i q^{\alpha_i}\rfloor/q^{\alpha_i})^{d-d'}} \le \frac{h(s_i)}{s_i^{d-d'}} \le (\frac{\lceil s_i q^{\alpha_i}\rceil}{\lceil s_i q^{\alpha_i}\rceil-1})^{d-d'}\frac{h(\lceil s_i q^{\alpha_i}\rceil/q^{\alpha_i})}{(\lceil s_i q^{\alpha_i}\rceil/q^{\alpha_i})^{d-d'}}$$
$$\implies (\frac{p^{n_0-1}}{p^{n_0-1}+1})^{d-d'}\frac{h(\lfloor s_i q^{\alpha_i}\rfloor/q^{\alpha_i})}{(\lfloor s_i q^{\alpha_i}\rfloor/q^{\alpha_i})^{d-d'}} \le \frac{h(s_i)}{s_i^{d-d'}} \le (\frac{p^{n_0-1}}{p^{n_0-1}-1})^{d-d'}\frac{h(\lceil s_i q^{\alpha_i}\rceil/q^{\alpha_i})}{(\lceil s_i q^{\alpha_i}\rceil/q^{\alpha_i})^{d-d'}}.$$

Let $i \to \infty$, then $s_i \to 0$, $\alpha_i \to \infty$. Since $\lfloor s_i q^{\alpha_i}\rfloor, \lceil s_i q^{\alpha_i}\rceil$ lies in $[p^{n_0-1}, p^{n_0}]$, so there are only finitely many possible values of $\lfloor s_i q^{\alpha_i}\rfloor, \lceil s_i q^{\alpha_i}\rceil$. So by Theorem 8.10,

$$\lim_{i\to\infty}\frac{h(\lfloor s_i q^{\alpha_i}\rfloor/q^{\alpha_i})}{(\lfloor s_i q^{\alpha_i}\rfloor/q^{\alpha_i})^{d-d'}} = \lim_{i\to\infty}\frac{h(\lceil s_i q^{\alpha_i}\rceil/q^{\alpha_i})}{(\lceil s_i q^{\alpha_i}\rceil/q^{\alpha_i})^{d-d'}} = c.$$

This means

$$(\frac{p^{n_0-1}}{p^{n_0-1}+1})^{d-d'}c \le \lim_{i\to\infty} h(s_i)/s_i^{d-d'} \le (\frac{p^{n_0-1}}{p^{n_0-1}-1})^{d-d'}c.$$

Since this is true for arbitrary n_0, we get

$$\lim_{i\to\infty} h(s_i)/s_i^{d-d'} = c.$$

This finishes the proof of (1).

(2) follows from (1).

(3) Since R is a domain and $I \neq 0$, $d' = \dim R/I < \dim R = d$, $d - d' \geq 1$. So the order of $h(s)$ at 0 is at least 1; in particular, $\lim_{s\to 0^+} h(s) = 0 = h(0)$. □

Lemma 8.12. *Let $(R, \mathfrak{m})$ be a noetherian local domain, I, J be two R-ideal such that $I + J$ is $\mathfrak{m}$-primary. Then $h_{R,I,J}(s)$ is continuous at 0 if and only if $I \neq 0$.*

Proof. If $I \neq 0$ then by previous theorem it is continuous at 0. If $I = 0$, then $h_R(s) = e_{HK}(J, R) \neq 0 = h_R(0)$ for $s > 0$, so it is discontinuous at 0. □

Theorem 8.13. *Let $(R, \mathfrak{m})$ be a noetherian local ring, I, J be two R-ideals such that $I + J$ is $\mathfrak{m}$-primary, M be a finitely generated R-module. Then $h_{M,I,J}(s)$ is continuous at 0 if and only if $I \not\subseteq P$ for any $P \in \operatorname{Supp}(M)$ with $\dim R/P = \dim M$. In particular, $h_{R,I,J}(s)$ is continuous at 0 if and only if $\dim R > \dim R/I$. If h_M is discontinuous at 0 then we have*

$$\lim_{s\to 0^+} h_M(s) = \sum_{P\in\operatorname{Supp}(M), I\subset P, \dim R/P=\dim M} l_{R_P}(M_P)e_{HK}(J, R/P).$$

Proof. By the associativity formula for h-function in Corollary 7.6,

$$h_M(s) = \sum_{P\in\operatorname{Supp}(M),\dim R/P=\dim M} l_{R_P}(M_P)h_{R/P}(s).$$

For any $P \in \operatorname{Supp}(M)$, $\lim\limits_{s\to 0+} h_{R/P,I,J}(s)$ is always non-negative; the limit is positive if and only if $I \subseteq P$, in which case the limit is $e_{HK}(J, R/P)$; see Lemma 8.12. Thus taking limit as s approaches zero from the right, we get the expression of the right hand limit of h_M. Since h_M is continuous at 0 if and only if $\lim_{s\to 0^+} h_{R/P}(s) = 0$ for any $P \in \operatorname{Supp}(M)$ with $\dim R/P = \dim M$, the continuity of h_M at zero is equivalent to asking $I \not\subseteq P$ for any $P \in \operatorname{Supp}(M)$ with $\dim R/P = \dim M$. If $M = R$, then this means $I \not\subseteq P$ for any $P \in \operatorname{Assh}(R)$ which means $\dim R > \dim R/I$. □

9. Questions

Inspired by Trivedi's question [Tri21, Question 2], we ask

Question 9.1. Let I, J be $\mathfrak{m}$-primary ideals of a noetherian local ring R. Is $h_{R,I,J}$ a piecewise polynomial? In other words, does there exists a countable subset S of $\mathbb{R}$ and a covering $\mathbb{R} \setminus S = \coprod_{n\in\mathbb{N}} (a_n, b_n)$ such that on each (a_n, b_n), $h_{R,I,J}$ is given by a polynomial function?

We point out that, in the context of the question, $h_{R,I,J}(s)$ is $e_{HK}(J, R)$ for large s, $e(I, R)s^{\dim(R)}/\dim(R)!$ on some interval $(0, a]$ and zero for s nonpositive.

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150 N. University Street West Lafayette, IN 47907-2067, United States
Email address: cheng319@purdue.edu

Institute of Mathematics, CAG, EPFL SB MATH MA A2 383 (Bâtiment MA), Station 8, CH-1015 Lausanne, Switzerland
Email address: alapan.mathematics@gmail.com